Universal Polar Codes for More Capable and Less Noisy Channels and Sources

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David Sutter, Joseph M. Renes

Institute for Theoretical Physics

ISIT 2014, Hawaii

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Motivation

Existence of optimal non-universal codes

Given two DMCs $W : \mathcal{X} \to \mathcal{Y}$ and $V : \mathcal{X} \to \mathcal{Z}$ with the same capacity-achieving input distribution and the same capacity. Does there exist a code that achieves the capacity of W but not of V, using **optimal decoding**?

- In general, it is desirable to have universal codes
- A non-universal capacity-achieving code could be beneficial for sending quantum information over a quantum channel at a rate > coherent information [Renes *et al.*'13]

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Canditate: polar codes

Notation & Definitions

- Given two DMCs $W : \mathcal{X} \to \mathcal{Y}$ and $V : \mathcal{X} \to \mathcal{Z}$
- X^n with X_i i.i.d. Bernoulli(p), $p \in [0, 1]$;
- $Y^n = W^n X^n$ and $Z^n = V^n X^n$
- $U^n = G_n X^n$ with $G_n := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)^{\log n}$

For $\varepsilon > 0$ consider the two low-entropy sets

- $\mathcal{D}^n_\varepsilon(X|Y) := \left\{ i \in [n] : H\left(U_i \middle| U^{i-1}, Y^n \right) \leq \varepsilon \right\}$
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Definition: degraded

- V is a (stochastically) degraded version of W if $\exists\ T : \mathcal{Y} \to \mathcal{Z}$ s.t. $V(z|x) = \sum_{y \in \mathcal{Y}} W(y|x) T(z|y)\ \forall x \in \mathcal{X}, z \in \mathcal{Z}$

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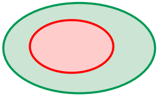

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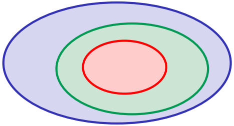

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- W is more capable than V if $I(X; Y) \ge I(X; Z)\ \forall P_X$

Universality of Polar Codes — History & Contribution

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Badly understood ☹

Relation between $\mathcal{D}_\varepsilon^n(X|Y)$ and $\mathcal{D}_\varepsilon^n(X|Z)$?

Would be extremely helpful for

- code construction
  - BEC is easy
  - channel up/downgrading [Tal-Vardy'11];
- network coding tasks
  - wiretap channel [Mahdavifar-Vardy'11, Şaşoğlu-Vardy'13]
  - broadcast channel [Goela *et al.*'13]
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Good news ☺

For specific classes of channels a few things are known

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Polar codes are universal for less noisy channels

Theorem: universality for less noisy channels

Let $W : \mathcal{X} \to \mathcal{Y}$ and $V : \mathcal{X} \to \mathcal{Z}$ be two DMCs such that W is less noisy than V. Then for any $\varepsilon \in (0,1)$, $n = 2^k$, $k \in \mathbb{N}$ we have $\mathcal{D}_\varepsilon^n(X|Z) \subseteq \mathcal{D}_\varepsilon^n(X|Y)$.

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- Let V and W be symmetric. Every polar code built for V can be used for W with SC decoding
- Recall that the class of less noisy channels is strictly larger than the class of degradable channels

Example: BEC – BSC pair [El Gamal-Kim'11]

Let $W = \mathrm{BEC}(\alpha)$ for $\alpha \in (0, \frac{1}{2})$ and $V = \mathrm{BSC}(\beta)$. Then

- $0 < \alpha \leq 2\beta$: V is a degraded w.r.t. W
- $2\beta < \alpha \leq 4\beta(1-\beta)$: W is less noisy than V

Proof Sketch

To show: $\mathcal{D}_\varepsilon^n(X|Z) \subseteq \mathcal{D}_\varepsilon^n(X|Y)$

Lemma 1: [thanks to Chandra Nair]

Let W and V be two DMCs such that W is less noisy than V. Then, W^n is less noisy than V^n for all $n \in \mathbb{N}$.

Lemma 2: [Csiszár-Körner'78]

Let $W : \mathcal{X} \to \mathcal{Y}$ and $V : \mathcal{X} \to \mathcal{Z}$ be two DMCs s.t. W is more capable than V. Then $I(X;Y|U) \geq I(X;Z|U)$ $\forall$ $P_{U,X}$, where $U \multimap X \multimap (Y,Z)$.

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Recall that

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Lemma 1 implies $H(U_1|Y^n) \leq H(U_1|Z^n)$
To show: $H\left(U_i \middle| U^{i-1}, Y^n\right) \leq H\left(U_i \middle| U^{i-1}, Z^n\right)$ for $2 \leq i \leq n$

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Consider the Markov chain $U^{i-1} -\!\circ\!- U^i -\!\circ\!- X^n -\!\circ\!- (Y^n, Z^n)$

$$
\begin{aligned}
H\left(U_i \middle| U^{i-1}, Y^n\right) &= H\left(U^i \middle| U^{i-1}, Y^n\right) \\
&\le H\left(U^i \middle| U^{i-1}, Z^n\right) \\
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Lemma 1 & Lemma 2

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Theorem: universality for more capable channels

Let P_X be such that it maximizes $I(X;Y) - I(X;Z)$. Then for $\varepsilon = O(2^{-n^\beta})$ with $\beta < \frac{1}{2}$, we have $\mathcal{D}_\varepsilon^n(X|Z) \dot{\subseteq} \mathcal{D}_\varepsilon^n(X|Y)$.

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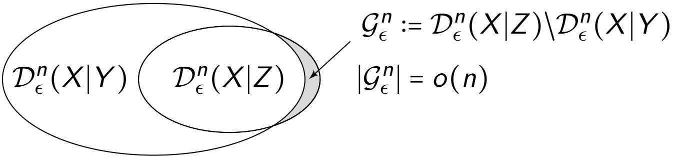

Summary & Outlook

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- Polar codes are universal for less noisy (symmetric) channels
- For a specific input distribution, polar codes are universal for more capable channels
- Can this be useful for code construction?
- This new insights might be useful for multi-terminal coding tasks
  - wiretap channel [Mahdavifar-Vardy'11, Şaşoğlu-Vardy'13]
  - broadcast channel [Goela *et al.*'13]
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