On the leading hadronic contribution to the muon (g−2) from μe−scattering

Amedeo Primo

ETH Zürich, 17th April 2018 – Particle Physics Seminar

Outline

- The muon (g-2) in the Standard Model
- Leading hadronic contribution from μe scattering
- μe scattering at NNLO in QED: virtuals
- Prospects

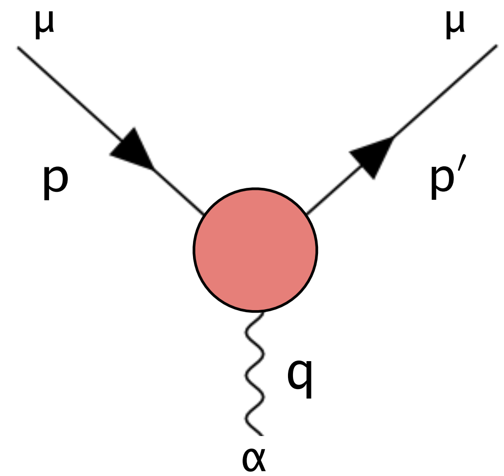
References: arXiv:1709.07435, 1704.05465, 1605.03157

In collaboration with: S. Di Vita, P.Mastrolia, M.Passera, T.Peraro, U.Schubert, L.Tancredi and W.J.Torres Bobadilla

The muon g-2

- Muon anomalous magnetic moment

$$\vec{m} = 2(1 + a_\mu)\frac{Qe}{2m_\mu}\vec{s}$$



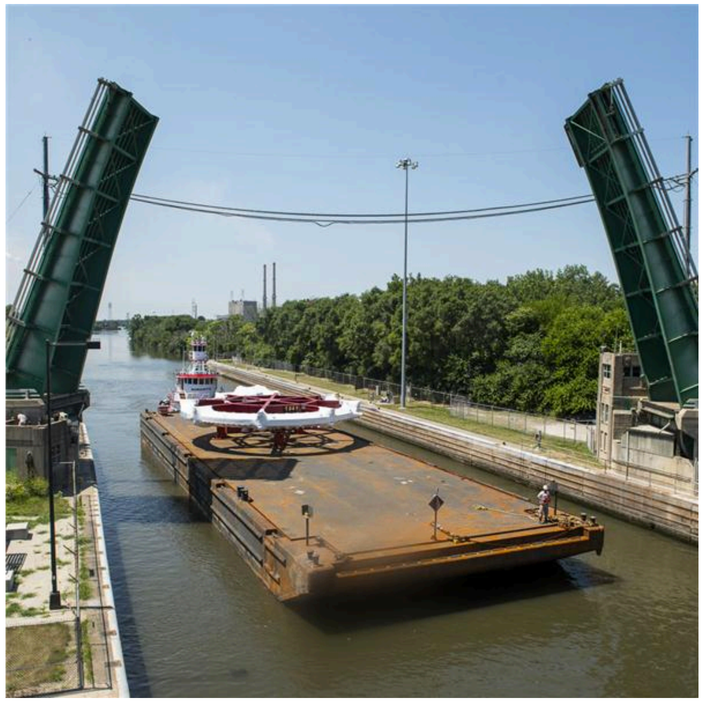
- Contributions from quantum effects, $a_\mu = F_2(0)$

$$\bar{u}(p')\Gamma_\alpha u(p) = \bar{u}(p')\left[\gamma_\alpha F_1(q^2) + \frac{i\sigma_{\alpha\beta}q^\beta}{2m_\mu^2}F_2(q^2) + \ldots\right]u(p)$$

- Experimental measure by BNL-E821, 0.5ppm accuracy

$$a_\mu^{exp} = 116\,592\,089\,(63) \times 10^{-11}$$

[E821 06]



- Upcoming validation with higher precision
  - FNAL-E989 aims at $\pm 16 \times 10^{-11}$ (0.14ppm)
  - Later confirmation from J-Parc E34

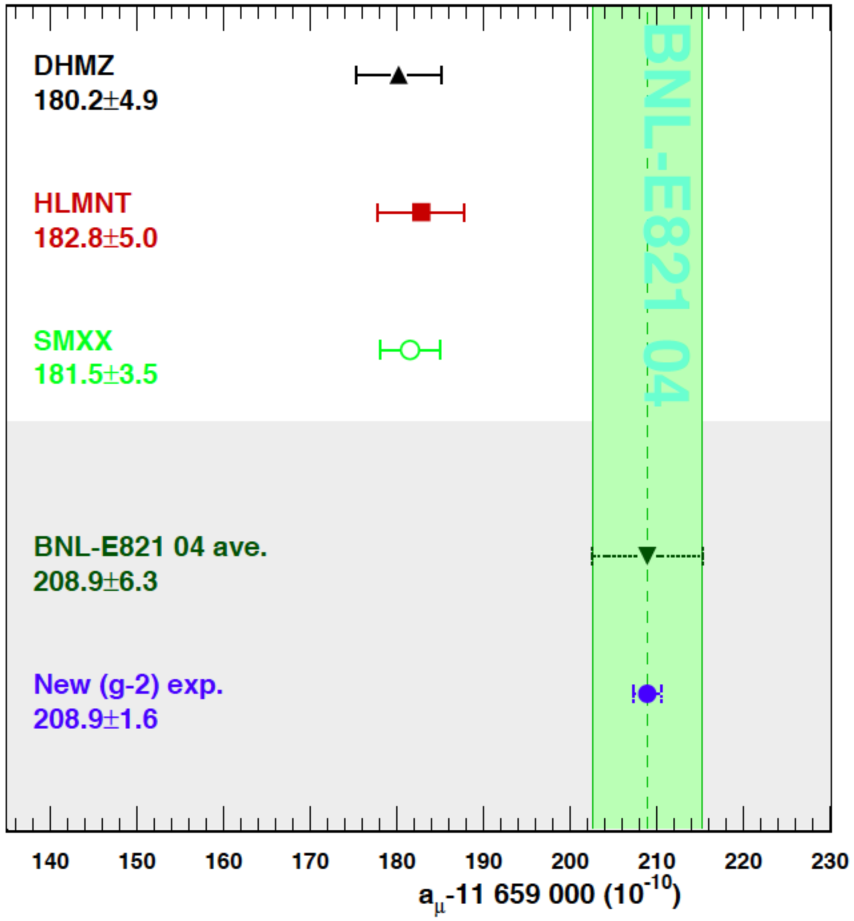
SM vs experiment

- (g-2) in the Standard Model

$$a_\mu = a_\mu^{QED} + a_\mu^{Weak} + a_\mu^{Had}$$

  - Theory prediction at 0.48 ppm accuracy

[Blum, Denig, Lovashenko et al 13]



- Longest standing deviation from the SM

$a_\mu^{SM} \times 10^{11}$	$\Delta a_\mu \times 10^{11}$	σ
116 591 761 (57)	330 (85)	3.9
116 591 818 (51)	273 (81)	3.4
116 591 841 (58)	250 (86)	2.9

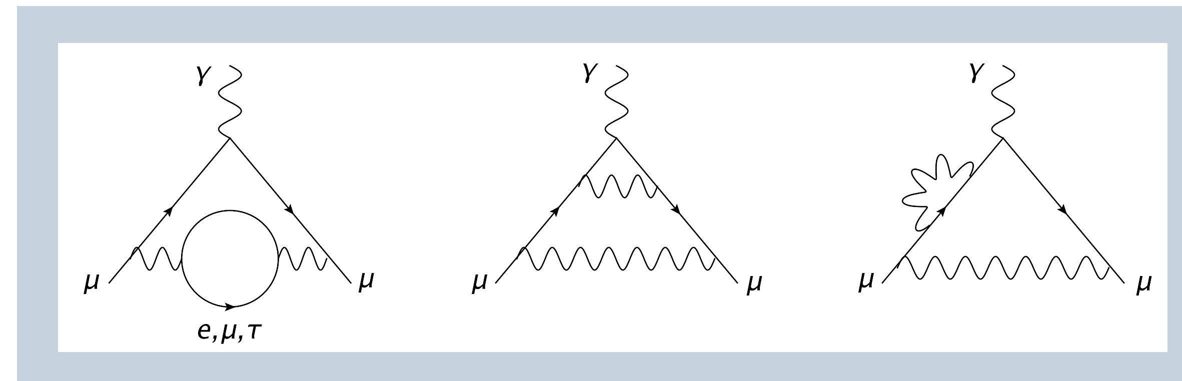
[Jegerlehner 15, Davier 16, Hagiwara et al 11]

  - New measurement can push σ above 5
  - Theoretical error will dominate

Electroweak contributions

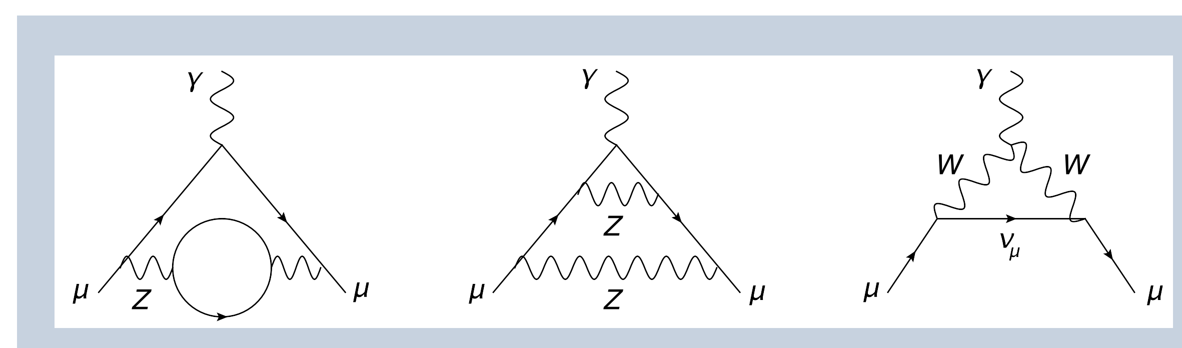
- EW sector is under complete control

- $$a_\mu^{QED} = 116\,584\,718.944(21)(77) \times 10^{-11}$$
  - 99.99% of the total
  - Known up to five loops
  - Uncertainty far below Δa_μ



[Schwinger 48,Sommerfield; Petermann; Suura andWichmann 57 Elend 66, Kinoshita and Lindquist 81, Kinoshita et al. 90, Remiddi, Laporta, Barbieri et al; Czarnecki and Skrzypek,Passera 04 Friot, Greynat and de Rafael 05, Mohr, Kinoshita & Nio 04-05, Aoyama, Hayakawa, Kinoshita et al 07, Taylor and Newell 12, Kinoshita et al. 12-15, Steinhauser et a 13-15-16, Yelkhovsky, Milstein, Starshenko, Laporta, Aoyama, Hayakawa, Kinoshita, Nio 12-15,Laporta 17,...]

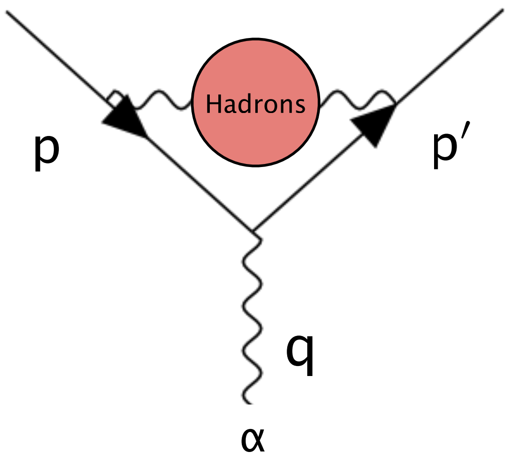
- $$a_\mu^{Weak} = (153.6 \pm 1) \times 10^{-11}$$
  - contributes to 1.5 ppm
  - known up to two loops
  - Uncertainty from hadronic loop



[Jackiv, Weinberg; Bars, Yoshimura; Altarelli, Cabibbo, Maiani; Bardeen, Gastmans, Lautrup; Fujikawa, Lee, Sanda 71, Kukhto et al. 92, Czarnecki, Krause, Marciano 95, Knecht, Peris, Perrottet, de Rafael 02, Czarnecki, Marciano and Vainshtein 02, Degrassi and Giudice 98; Heinemeyer, Stockinger, Weiglein (04), Gribouk and Czarnecki 05, Vainshtein 03, Gnendiger, Stockinger, Stockinger-Kim 13,...]

Hadronic contribution

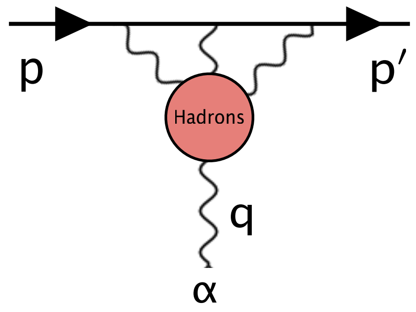
- Hadronic contribution: 60 ppm of the total
  - Non-perturbative, large uncertainties
- $\Pi_{Had}(q^2)$ largest contribution to central value



$$a_\mu^{HLO} = 6870\,(42) \times 10^{-11}$$
$$= 6926(33) \times 10^{-11}$$
$$= 6949(37)(21) \times 10^{-11}$$

[Jegerlehner 15, Davier 16, Hagiwara et al 11]

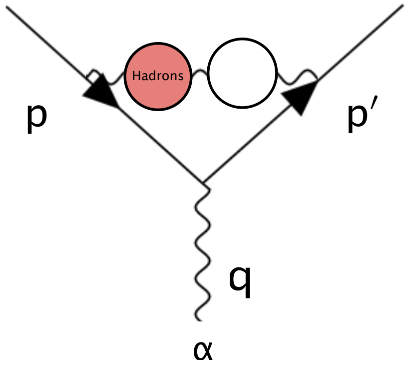
- Light-by-Light



$$a_\mu^{LBL} = 102\,(39) \times 10^{-11}$$

[Knecht, Nyffeler 02, Melnikov, Vainshtein 03...., Jegerlehner 15]

- Hadronic NLO



$$a_\mu^{NHLO} = -98\,(1) \times 10^{-11}$$

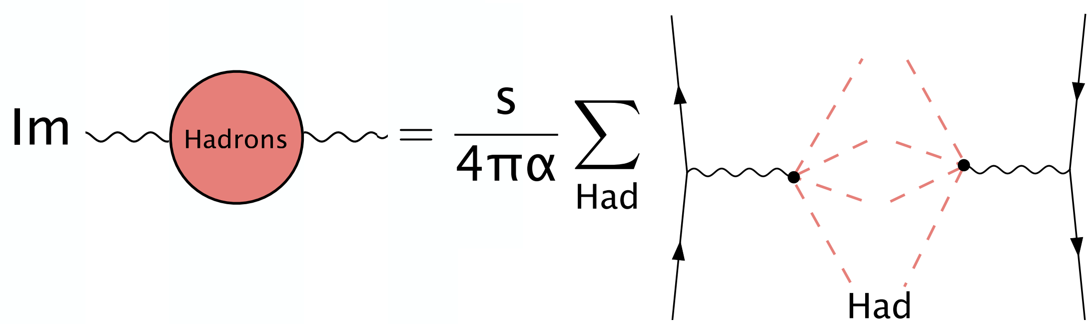
[Krause 96, Alemany et al 98,..., Hagiwara et al 11]

Dispersive approach to a_μ^{HLO}

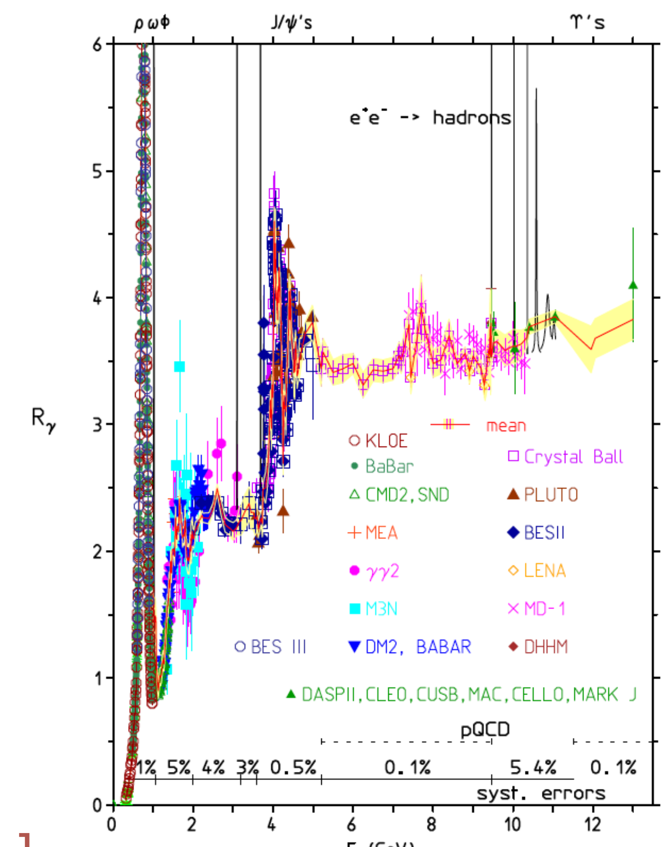
- a_μ^{HLO} computed from dispersion relations

$$a_\mu^{HLO} = \frac{\alpha}{\pi^2}\int_{4m_\pi^2}^{\infty}\frac{ds}{s}\int_0^1 dx\,\frac{x^2(1-x)}{x^2+(1-x)s/m^2}\,\mathrm{Im}\Pi(s)_{Had}$$

  - Unitarity relates $\mathrm{Im}\,\Pi_{had}(s)$ to $e^+e^- \to$ Had cross section



[Jegerlehner 15]



- Extract a_μ^{HLO} from experimental data

$$a_\mu^{HLO} = \frac{1}{4\pi^3}\int_{4m_\pi^2}^{\infty} ds\int_0^1 dx\,\frac{x^2(1-x)}{x^2+(1-x)s/m^2}\,\sigma_{e^+e^-\to Had}(s)$$

[Bouchiat, Michiel 61, Durand 62, Gourdin, de Rafael 69,...]

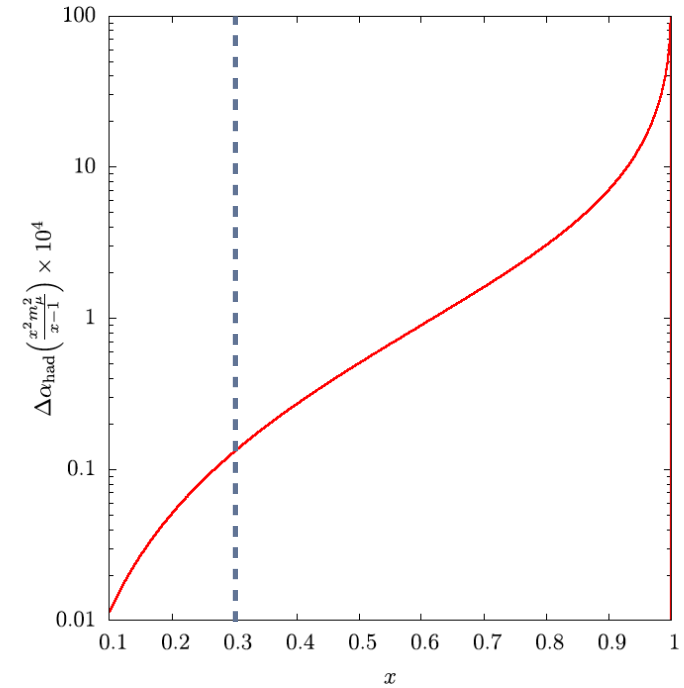
  - enhanced region $s \lesssim 2\,\mathrm{GeV}$
  - Improve accuracy to 0.22 ppm requires 0.4% error on $\sigma_{e^+e^-\to Had}$

a_μ^{HLO} from muon–electron scattering

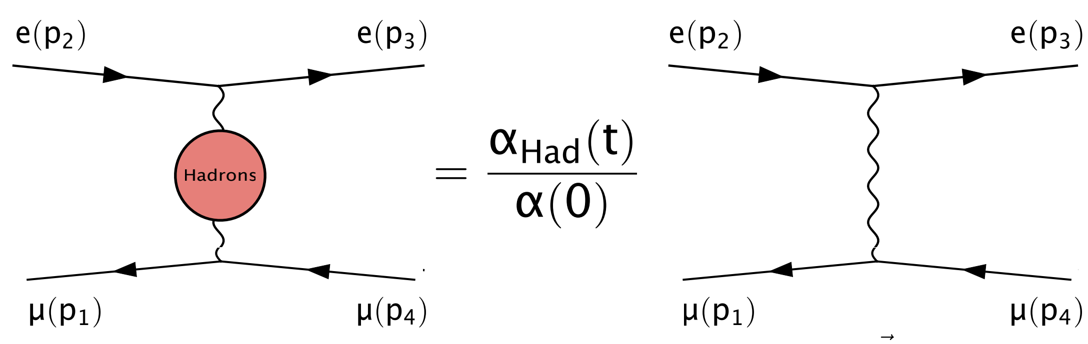
- Alternatively, compute a_μ^{HLO} from space-like data

$$a_\mu^{HLO} = \frac{\alpha}{\pi}\int_0^1 dx\,(1-x)\Delta\alpha_{Had}[t(x)] \qquad t(x) = \frac{x^2 m^2}{x-1} \leq 0$$

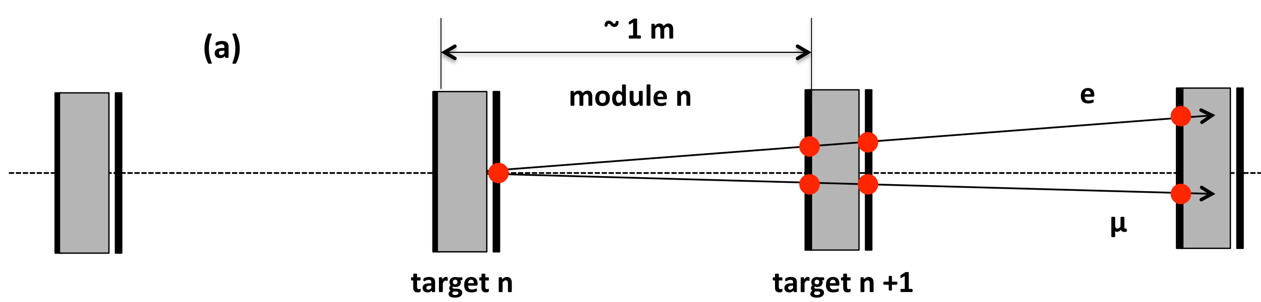
[Lautrup, Peterman, de Rafael 72]

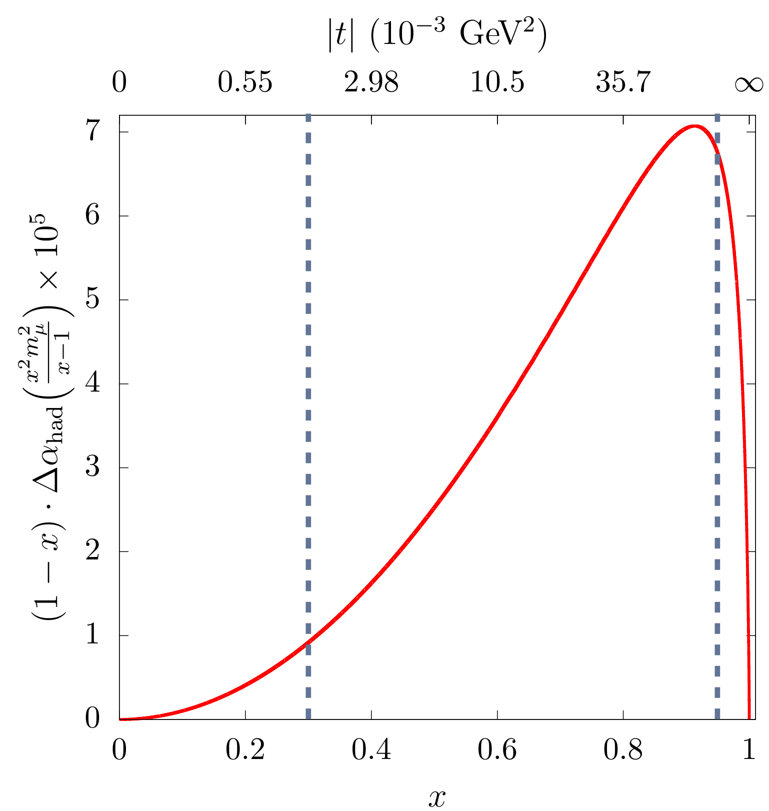


- Extract $\Delta\alpha_{Had}[t(x)] = -\overline{\Pi}_{Had}[t(x)]$ from μe scattering



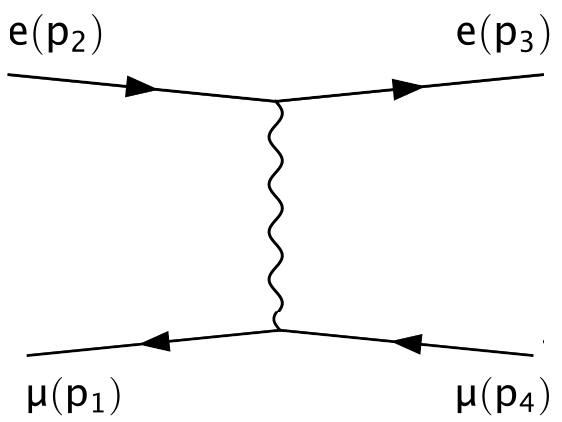
- MUonE proposal: 150 GeV μ-beam on Be layers





[Carloni Calame, Passera et al 15, Abbiendi, Carloni Calame, Marconi et al 16]

a_μ^{HLO} from muon–electron scattering



- Running coupling from μe scattering

$$\frac{d\sigma^{HLO}}{dt} = \left|\frac{\alpha_{Had}(t)}{\alpha(0)}\right|^2 \frac{d\sigma^{HLO}}{dt}$$

  - LO contribution from QED

$$\frac{d\sigma_{LO}}{dt} = 4\pi\alpha^2 \frac{(m^2+m_e^2) - su - t^2/s}{t^2\lambda(s,m^2,m_e^2)}$$

  - Kinematics $s = (p_1+p_2)^2, \quad t = (p_2-p_3)^2, \quad u = 2m^2 + 2m_e^2 - s - t$

- Measure the cross section, subtract everything but the hadronic vac pol

  - 20×10^{-11} estimated statistical uncertainty on a_μ^{HLO} (0.3%)
  - systematics (exp. and th.) must be below 10 ppm

- Theory goal: Monte Carlo for QED μe at NNLO

Muon–electron at NNLO

- Needed fixed order corrections to μe scattering

$$\sigma_{NLO} = \underbrace{\int dLIPS_2 \left(2\text{Re}\mathcal{M}^{(0)\,*}\mathcal{M}^{(1)}\right)}_{\text{Virtual}} + \underbrace{\int dLIPS_{3;} |\mathcal{M}_\gamma^{(0)}|^2}_{\text{Real}}$$

[Nikishov 61, Eriksson 61, ...]

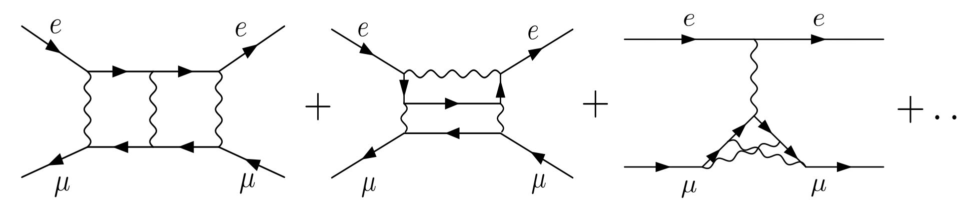
$$\sigma_{NNLO} = \underbrace{\int dLIPS_2 \left(2\text{Re}\,\mathcal{M}^{(0)*}\mathcal{M}^{(2)} + |\mathcal{M}^{(1)}|^2\right)}_{\text{Double Virtual}} + \underbrace{\int dLIPS_3 2\text{Re}\,\mathcal{M}_\gamma^{(0)*}\mathcal{M}_\gamma^{(1)}}_{\text{Real–Virtual}} + \underbrace{\int dLIPS_4 |\mathcal{M}_{\gamma\gamma}^{(0)}|^2}_{\text{Double Real}}$$

- $\mathcal{M}^{(2)}(m_e^2, m^2)$ unknown and out of reach
  - Given $m_e^2/m^2 \approx 2 \cdot 10^{-5}$, consider massless electron
  - $\mathcal{M}^{(2)}(0, m^2)$ can be computed but need to recover $\alpha^2 \ln^n(m_e^2/m^2)$

Workflow

Amplitude generation

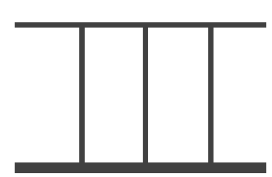
- $\mathcal{M}^{(2)}(e\mu \to e\mu) =$



Algebraic decomposition

- $$\mathcal{M}^{(2)}(e\mu \to e\mu) = \sum_k c_k(s,t,m^2,\epsilon) I_k^{(2)}(s,t,m^2,\epsilon)$$
  - c_k rational coefficients (include tensor structure)
  - $I_k^{(2)} = \int d^d q_1 d^q q_2 \frac{1}{D_1^{a_1} \cdots D_{n_k}^{a_{n_k}}}, \qquad D_j = l_j^2(q_i, p_k) - m_j^2$

Loop integral evaluation

- 

$$= -\frac{4}{3t^2\epsilon^4\left(m^2-s\right)} + \frac{14G(1;\frac{s}{m^2}) - 9G\left(0;\frac{2m^2-t-\sqrt{t(t-4m^2)}}{2m^2}\right) + 18G\left(1;\frac{2m^2-t-\sqrt{t(t-4m^2)}}{2m^2}\right)}{6t^2\epsilon^3\left(m^2-s\right)} + \mathcal{O}(\epsilon^{-2})$$

Renormalisation, subtractions, ps-integration

Amplitude decomposition

Amplitude generation

Algebraic decomposition

Loop integral evaluation

Renormalisation,
subtractions, pp integration

- Form factor decomposition from Lorentz (+gauge) symmetry

$$\mathcal{M}(p_i \to p_f) = \mathcal{M}_{\mu_1 \cdots \mu_n}(p_i \to p_f) \prod_{j=1}^{n} \epsilon_j^{\mu_j}$$
$$= \left(\sum_k \mathcal{A}_k(p_i \to p_f) T^k_{\mu_1 \cdots \mu_n} \right) \prod_{j=1}^{n} \epsilon_j^{\mu_j}$$

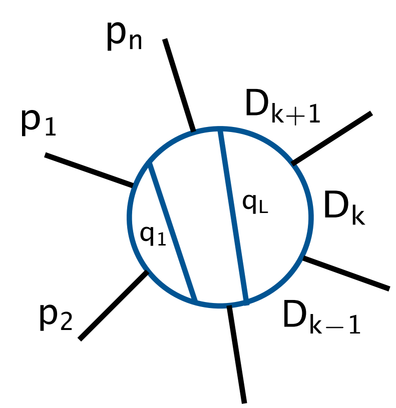
  - Extract $\mathcal{A}_k$ by projection

$$\mathcal{A}_k(p_i \to p_f) = \sum_{\text{spin}} P_k^{\mu_1 \cdots \mu_n}(p_i \to p_f) \mathcal{M}_{\mu_1 \cdots \mu_n}(p_i \to p_f)$$

- No analytic (integral) information required

  - Work at the integrand level [Ossola, Papadopoulos, Pittau 06]

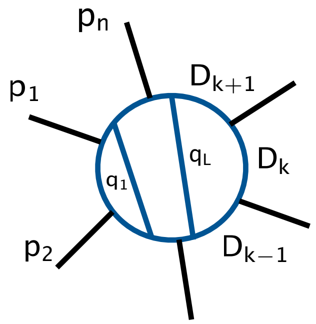
$$\mathcal{M}(p_i \to p_f) \longrightarrow$$



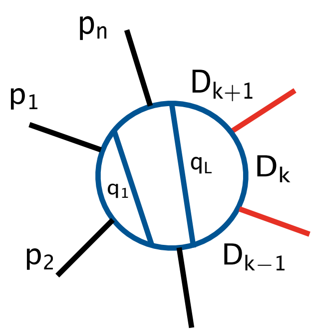
$$= \frac{N_{i_1 \cdots i_k \cdots i_m}(q_j)}{D_1 \cdots D_k \dots D_m}$$

The integrand decomposition method

- Algorithm: recursive partial fractioning of the integrand

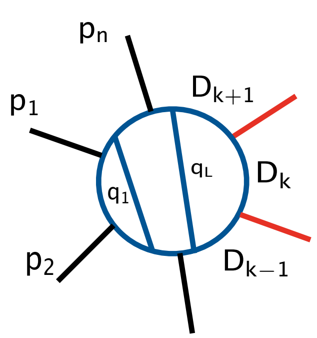


$$= \frac{N_{i_1\cdots i_k\cdots i_m}(q_j)}{D_1\cdots D_k\ldots D_m}$$

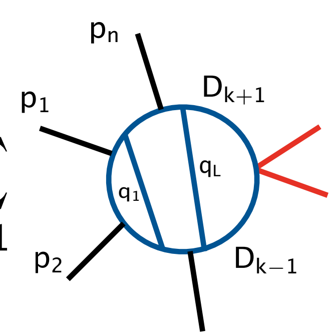


$$= \sum_{k=1}^{n} \frac{N_{i_1\cdots i_{k-1} i_{k+1}\cdots i_m}(q_j)\cancel{D_k}}{D_1\cdots \cancel{D_k}\ldots D_m} + \frac{\Delta_{i_1\cdots i_k\cdots i_m}(q_j)}{D_1\cdots D_k\ldots D_m}$$

$$\Delta_{i_1\cdots i_k\cdots i_m}(q_j) = N_{i_1\cdots i_k\cdots i_m}(q_j)\big|_{D_{i_k}=0}$$

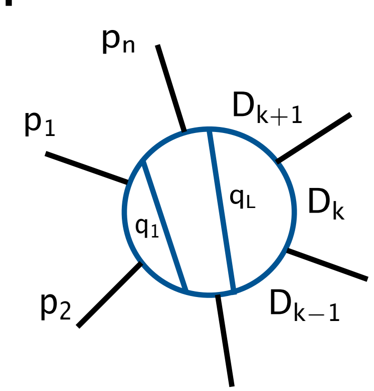


$$= \sum_{k=1}^{n}$$



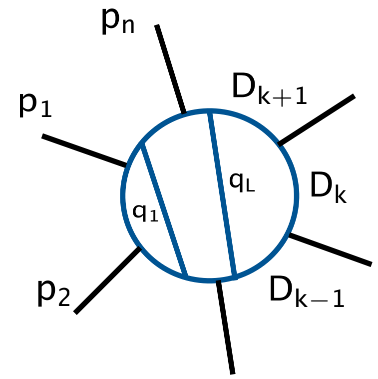
$$+ \frac{\Delta_{i_1\cdots i_k\cdots i_m}(q_j)}{D_1\cdots D_k\ldots D_m}$$

- Result: amplitude decomposed into all possible multi-particle cuts



$$= \sum_{\vec{b}_k \leq \vec{a}_k} \frac{\Delta_{\vec{b}_k}(q_j)}{D_1^{b_1}\cdots D_m^{b_m}}$$

Integrand decomposition



$$= \sum_{\vec{b}_k \leq \vec{a}_k} \frac{\Delta_{\vec{b}_k}(q_j)}{D_1^{b_1} \cdots D_m^{b_m}}$$

- Pros: very general approach to amplitude decomposition
  - Applicable to any theory
  - Extendible to arbitrary masses and external particles
  - Works in dimensional regularisation
  - Largely automatable [Ossola, Papadopoulos, Pittau 08, Mastrolia Ossola, Reiter, Tramontano 10, … , Peraro 14]
  - Led the NLO revolution [Hahn, Perez-Victoria 99, … ,Berger, Bern, Dixon et al 08, Hirschi, Frederix, Frixione, Garzelli 11, Cullen, Greiner, Heinrich Luisoni, Mastrolia 11, Cascioli, Maierhofer, Pozzorini 11,…]
- Cons: mathematical subtleties hide technical issues at higher loops
  - Multivariate polynomial division is not a standard division
  - Spurious terms generate large intermediate expressions

Polynomial division

- Division of $N(q_j)$ modulo $(D_1(q_j), D_2(q_j) \ldots, D_m(q_j))$

$$N(q_j) = \sum_{i=1}^{m} N_i(q_j) D_i(q_j) + \Delta(q_j) \qquad \{q_i\} = (x, y, z, \ldots)$$

  - Monomial ordering of the loop variables $x^2 y \overset{?}{\lessgtr} xy^2$
  - Divisor ordering $D_1 \to D_2 \to \cdots \to D_m \neq D_2 \to D_1 \to \cdots \to D_m$

- Find a Gröbner basis $(G_1(q_j), G_2(q_j), \ldots G_n(q_j))$ such that

$$\langle D_1 \cdots D_m \rangle = \left\{ P(q_i) = \sum_k p_i(q_i) D_k(q_i) \right\} = \left\{ P(q_i) = \sum_k q_i(q_i) G_k(q_i) \right\}$$

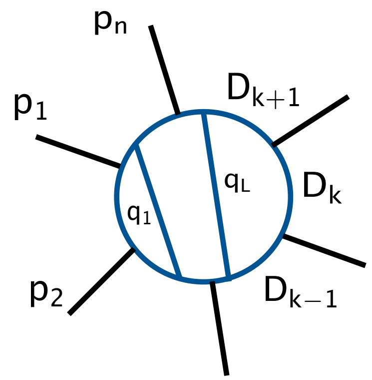
  - $\Delta(q_j)$ is univocally determined [Mastrolia, Ossola 11, Zhang 12-16, Badger, Frellesvig, Zhag 12-13, Mastrolia, Mirabella, Ossola, Peraro 12,...]
  - Inversion relation are needed

$$\sum_{i=1}^{m} N_i(q_j) D_i(q_j) = \sum_{i=1}^{n} Q_i(q_j) G_i(q_j)$$

Spurious terms

- The physical amplitude is integrated over the loop momenta

$$\int d^dq_1 \cdots d^dq_L$$



$$= \sum_{\vec{b}_k \leq \vec{a}_k} \int d^dq_1 \cdots d^dq_L \frac{\Delta_{\vec{b}_k}(q_j)}{D_1^{b_1} \cdots D_m^{b_m}}$$

- $\Delta_{\vec{b}_k}(q_j)$ contains spurious terms, $\Delta_{\vec{b}_k}(q_j) = \Delta^{NSP}_{\vec{b}_k}(q_j) + \Delta^{SP}_{\vec{b}_k}(q_j)$

$$\int d^dq_1 \cdots d^dq_L \frac{\Delta^{SP}_{\vec{b}_k}(q_j)}{D_1^{b_1} \cdots D_m^{b_m}} = 0$$

  - Simple reason, symmetry

$$\int d^dq \frac{q \cdot p}{(q^2 - m^2)^a} = 0$$

  - Symmetries of $\Delta_{\vec{b}_k}(q_j)$ can be hard to find

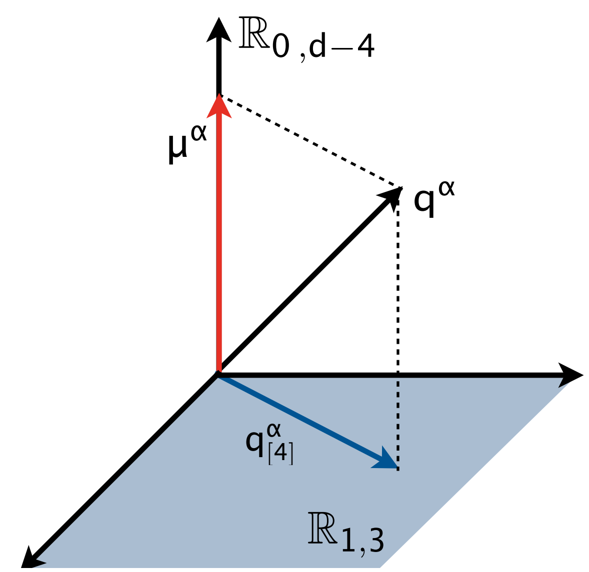
- The right choice of variables makes the difference

Dimensional regularisation

- Divergent loop integrals are dimensionally regularised, $\mathbb{R}_{1,3} \to \mathbb{R}_{1,d-1}$

$$\int d^4q \frac{N(q)}{D_1(q)\cdots D_m(q)} \to \int d^{d}q \frac{N(q)}{D_1(q)\cdots D_m(q)}$$

  - External particles kept in four dimensions, $p_i^\alpha, \epsilon_i^\alpha(p) \in \mathbb{R}_{1,3}$



$$d = 4 - 2\varepsilon \qquad\qquad q^\alpha = q_{[4]}^\alpha + \mu^\alpha$$

  - Integrands preserve spherical symmetry of $\mathbb{R}_{0,d-4}$

$$q_{[4]}^\alpha \to \vec{x} = (x_1, x_2, x_3, x_4) \qquad\qquad \mu^\alpha \to \mu^2$$

$$\int d^dq \frac{N(q)}{D_1(q)\cdots D_m(q)} \to \frac{1}{2}\int d^4q \int_0^\infty d\mu^2 (\mu^2)^{\frac{d-6}{2}} \int d\Omega_{d-4} \frac{N(\vec{x},\mu^2)}{D_1(\vec{x},\mu^2)\cdots D_m(\vec{x},\mu^2)}$$

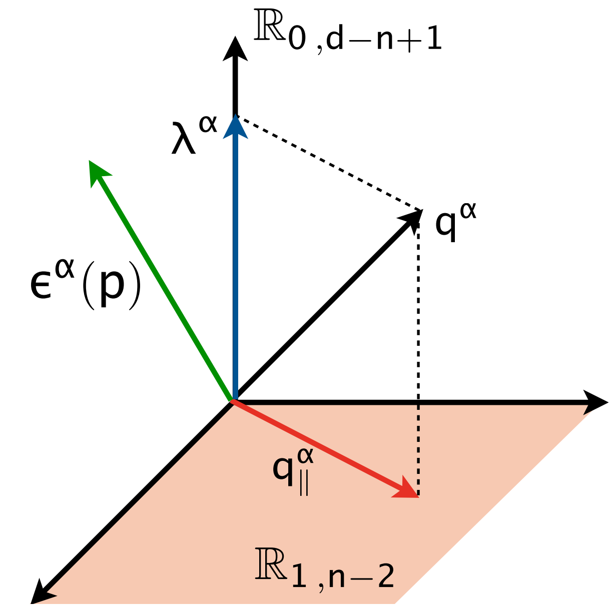
- Multiloop extension

$$\int d^dq_1 \cdots d^dq_L \to \int d^4q_{[4]\,1} \cdots d^4q_{[4]\,L} \int \prod_{i\leq j}^{L} \left[\det(\mu_i \cdot \mu_j)\right]^{\frac{d-5-L}{2}} \int \prod_{k=1}^{L} d\Omega_{d-3-k}$$

Transverse and longitudinal space

- Momentum conservation forces $\mathrm{Span}(p_1, \ldots, p_n) \subseteq \mathbb{R}_{1,3}$

$$d = d_{\parallel} + d_{\perp} \qquad q^{\alpha} = q^{\alpha}_{\parallel} + \lambda^{\alpha}$$



[Mastrolia, Peraro, **AP** 16]

- Numerators break spherical symmetry of $\mathbb{R}_{0,d-n+1}$

$$q^{\alpha}_{\parallel} \to \vec{x}_{\parallel} = (x_1, x_2, \ldots, x_{n-1})$$

$$\lambda^{\alpha} \to \{\vec{x}_{\perp} = (x_n, x_{n+1}, \ldots, x_4), \lambda^2\}$$

$$\int d^d q \frac{N(q)}{D_1(q)\cdots D_m(q)} \to \frac{1}{2}\int d^{n-1}q_{\parallel}\int_0^{\infty} d\lambda^2 (\lambda^2)^{\frac{d-n-1}{2}} \int d\Omega_{d-n+1} \frac{N(\vec{x}_{\parallel}, \vec{x}_{\perp}, \lambda^2)}{D_1(\vec{x}_{\parallel}, \lambda^2)\cdots D_m(\vec{x}_{\parallel}, \lambda^2)}$$

- Remove antisymmetric terms by expanding into orthogonal polynomials

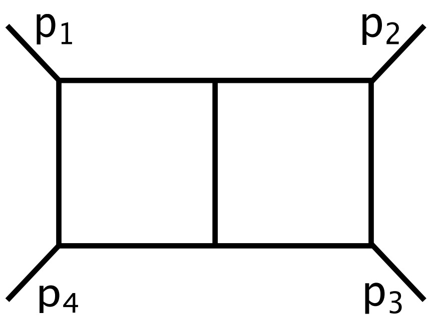
$$\int_{-1}^{1} dz\,(1-z^2)^{a-1/2}\, C^{(a)}_n(z) C^{(a)}_m(z) = \frac{\pi 2^{1-2a}\Gamma(n+2a)\delta_{nm}}{n!(n+a)\Gamma^2(a)}$$

- Multiloop extension

$$\int d^d q_1 \cdots d^d q_L \to \int d^{n-1}q_{\parallel\,1}\cdots d^{n-1}q_{\parallel\,L} \int \prod_{i\leq j}^{L} [\det(\lambda_i \cdot \lambda_j)]^{\frac{d-n-L}{2}} \int \prod_{k=1}^{L} d\Omega_{d-n+2-k}$$

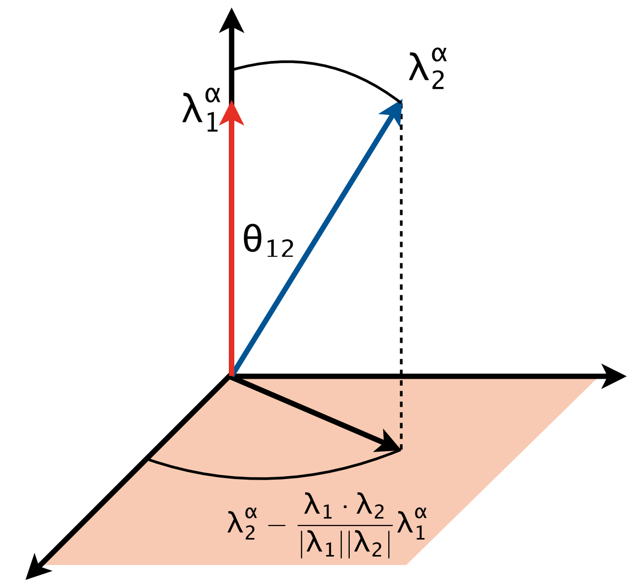
Two loop box integrals

- Four external particles, $d_{\parallel} = \dim\left[\mathrm{Span}(p_1, p_2, p_3, p_4)\right] = 3$



$$q^{\alpha}_{\parallel\, i} \to \vec{x}_{\parallel} = (x_{1i}, x_{2i}, x_{3i})$$

$$\{\lambda_1^{\alpha}, \lambda_2^{\alpha}\} \to \{\vec{x}_{\perp\, i} = x_{4i}, \lambda_{11}, \lambda_{12}, \lambda_{22}\}$$



  - Define transverse space

$$x_{41} = \sqrt{\lambda_{11}}\, c_{\theta_{11}}$$

$$x_{42} = \frac{1}{\sqrt{\lambda_{11}}}\left(\lambda_{12}\, c_{\theta_{11}} + (\lambda_{11}\lambda_{22} - \lambda_{12}^2)^{1/2}\, c_{\theta_{22}}\right)$$

  - Move to spherical coordinates

$$\int d^d q_1 d^d q_2 = \frac{2^{d-6}}{\pi^5 \Gamma(d-5)} \int d^3 q_{\parallel\,1} d^3 q_{\parallel\,2} \int d\lambda_{11} d\lambda_{12} d\lambda_{22} \left[\det(\lambda_i \cdot \lambda_j)\right]^{\frac{d-6}{2}} \int_{-1}^{1} dc_{\theta_{11}} dc_{\theta_{22}} s_{\theta_{11}}^{(d-6)} s_{\theta_{22}}^{(d-7)}$$

  - Expand in Gegenbauer polynomials of $c_{\theta_{11}}$ and $c_{\theta_{22}}$

$$\int d^d q_1 d^d q_2 \frac{x_{42}}{D_1 \cdots D_7} = 0$$

$$\int d^d q_1 d^d q_2 \frac{x_{41} x_{42}}{D_1 \cdots D_7} = \frac{1}{(d-3)} \int d^d q_1 d^d q_2 \frac{\lambda_{12}}{D_1 \cdots D_7}$$

$$\int d^d q_1 d^d q_2 \frac{x_{41}^3 x_{42}}{D_1 \cdots D_7} = \frac{3}{(d-3)(d-1)} \int d^d q_1 d^d q_2 \frac{\lambda_{12}\lambda_{11}}{D_1 \cdots D_7}$$

Adaptive Integrand Decomposition

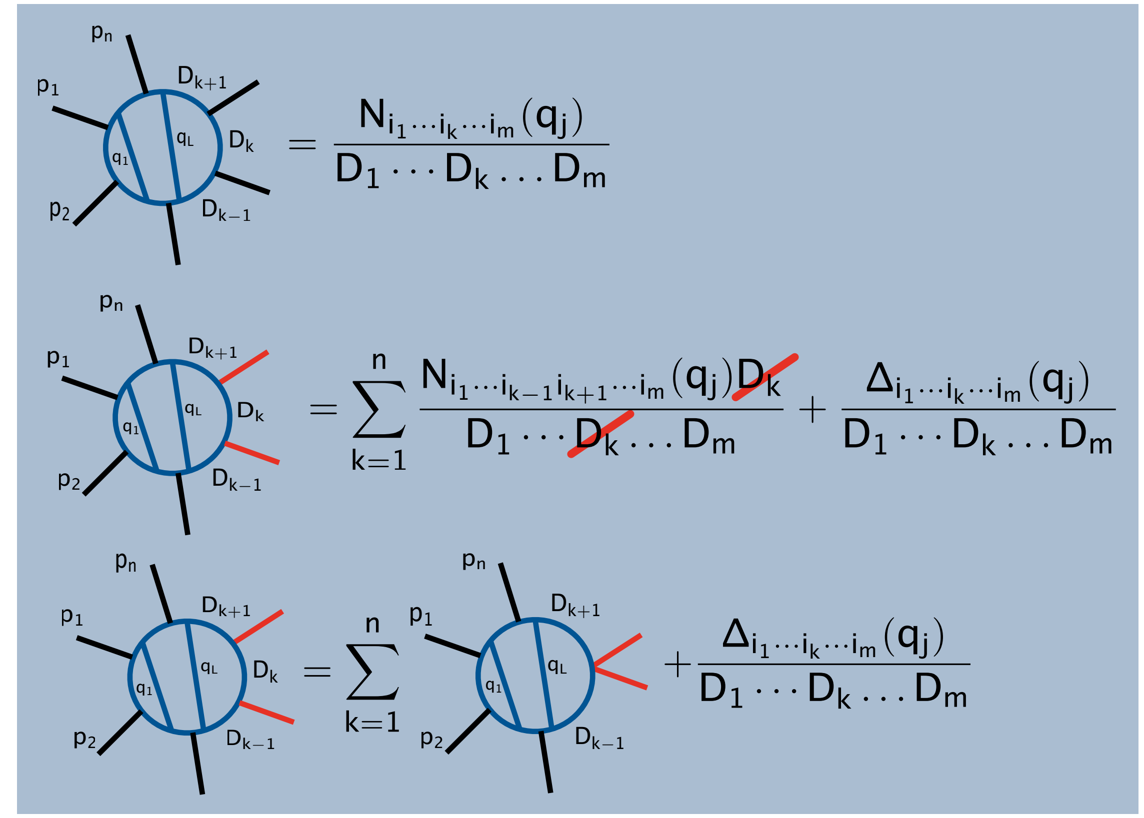
- At every step:
  - Parametrise q_i according to $\mathrm{Span}(p_1, \ldots, p_n)$

$$\{q_i^\alpha\} \to \{\vec{x}_{\parallel\, i}, \vec{x}_{\perp\, i}, \lambda_{ij}\}$$

  - Solve linear equations with $\lambda_{ij} \prec \vec{x}_{\parallel}$

$$L_k(\vec{x}_{\parallel}, \lambda_{ij}) = D_k$$

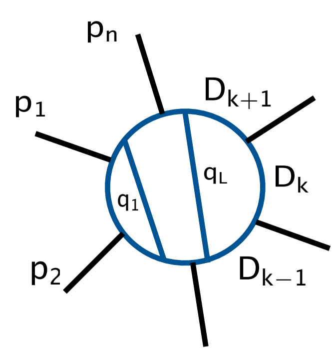
  - Identify $\Delta(q_i) = N(q_i)\Big|_{L_k(\vec{x}_{\parallel}, \lambda_{ij})=0}$



[Mastrolia, Peraro, AP 16]

- Final decomposition:

$$\int d^d q_1 \cdots d^d q_L$$



$$= \sum_{\vec{b}_k \le \vec{a}_k} \int d^d q_1 \cdots d^d q_L \frac{\Delta^{NSP}_{\vec{b}_k}(\vec{x}_{\parallel})}{D_1^{b_1} \cdots D_m^{b_m}}$$

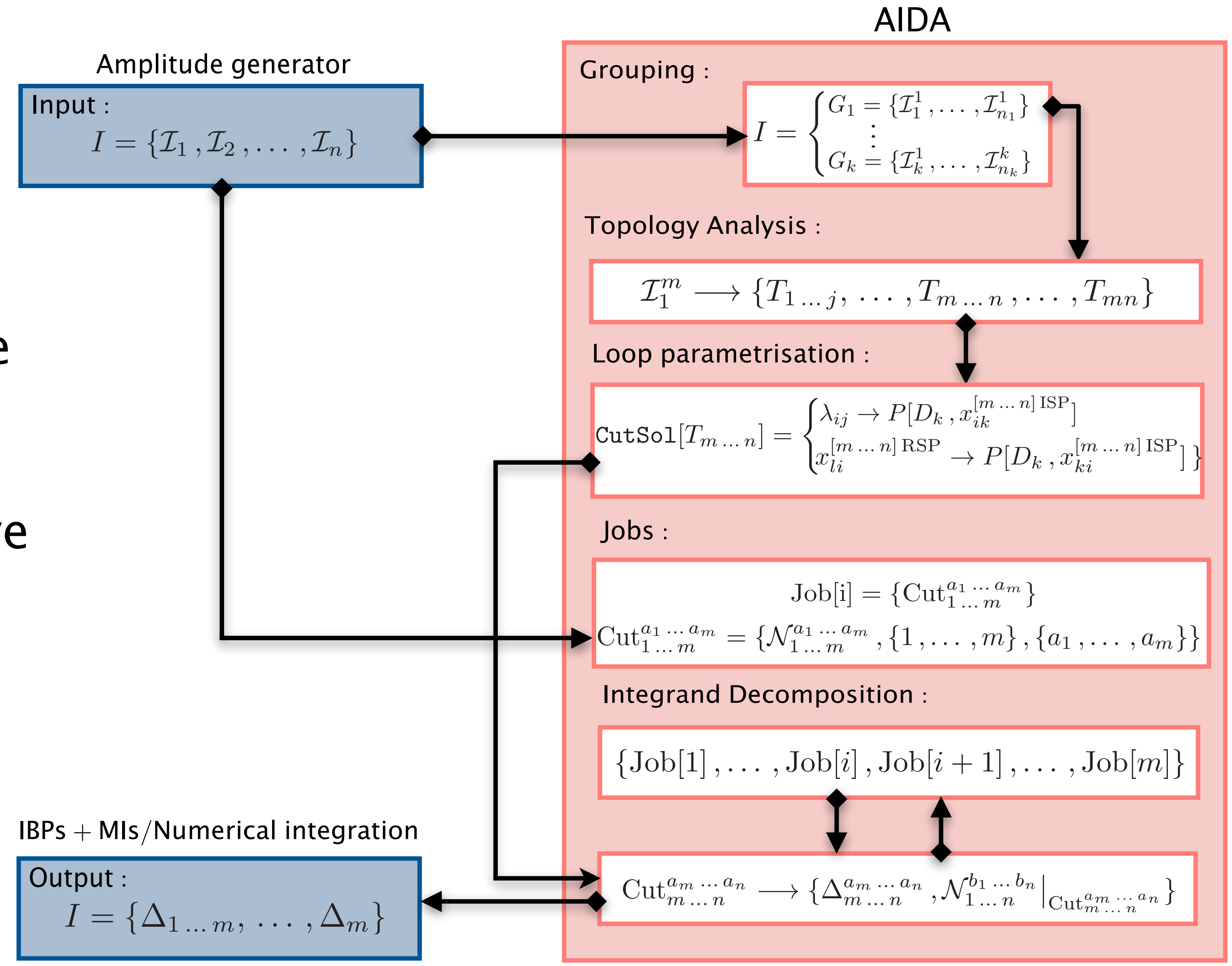
  - No computation of Gröbner basis
  - No spurious terms
  - Amplitude decomposed into scalar integrals

Implementation

- Mathematica implementation

  [Mastrolia, Peraro, AP, Torres Bobadilla, Ronca]

  - Input: one- or two- loop amplitude
  - Output: analytic/numerical adaptive integrand decomposition
  - Interfaced with IBPs reduction and sector decomposition programs



process	scales(+d)	$\mathcal{C}$	Analytic	Numeric
$e\mu \to e\mu$	s, t, m	9	✓	✓
$gg \to Hg$	s, t, m_t, m_h	10	✓	✓
$gg \to HH$	s, t, m_t, m_h	10	✓	✓
$gg \to ggg$	$s_{12}, s_{23}, s_{34}, s_{45}, s_{51}$	12	x	✓
$gg \to Hgg$	$s_{12}, s_{23}, s_{34}, s_{45}, s_{51}, m_t, m_h$	12	x	✓

Loop integrals

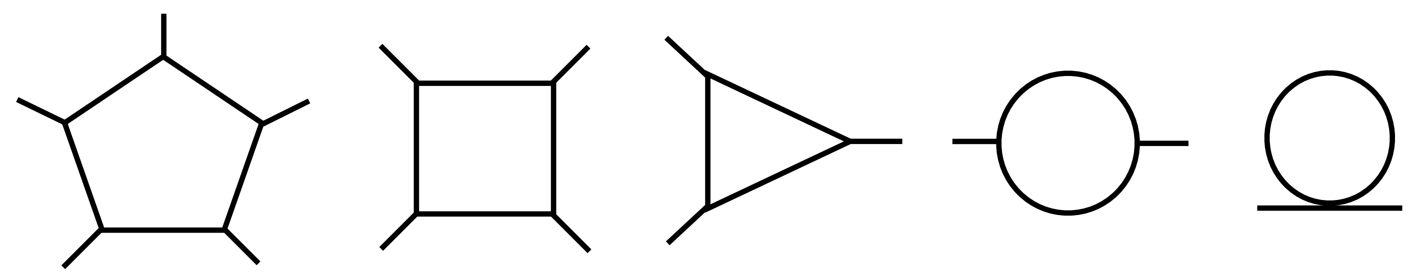
Amplitude generation

Algebraic decomposition

Loop integral evaluation

Renormalisation, subtractions, pp integration

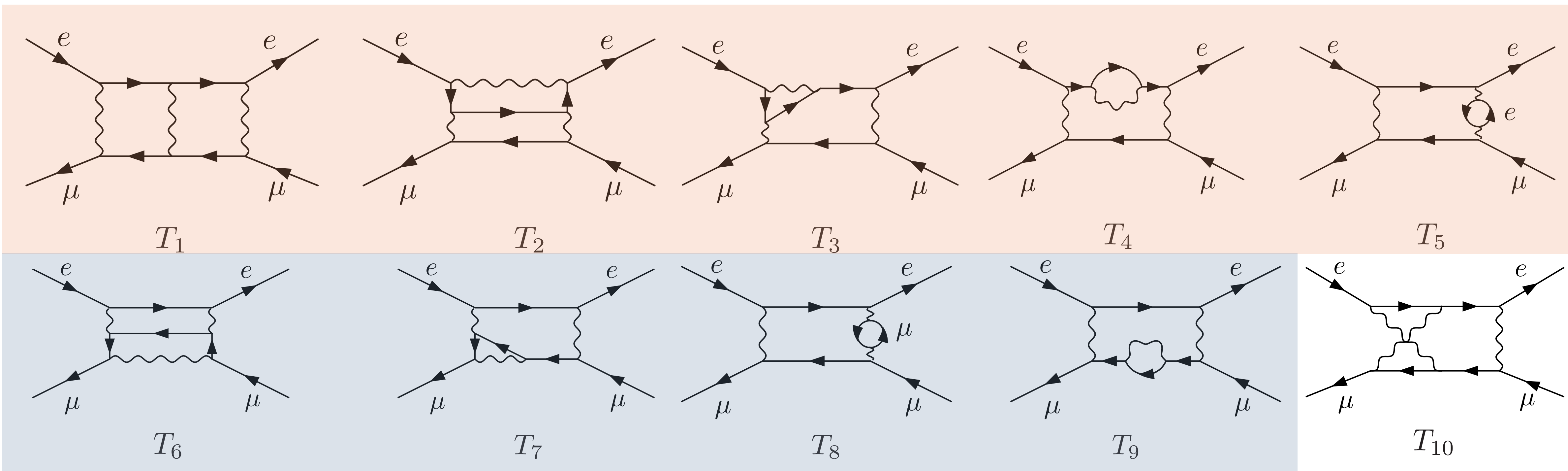
- $\mathcal{M}^{(2)}(e\mu \to e\mu) = \sum_k c_k(s,t,m^2,\epsilon) I_k^{(2)}(s,t,m^2,\epsilon)$
  - c_k rational coefficients
  - $I_k^{(2)} = \int d^dq_1 d^qq_2 \frac{1}{D_1^{a_1}\cdots D_{n_k}^{a_{n_k}}}\,, \qquad D_j = l_j^2(q_i,p_k) - m_j^2$
- One-loop: small number of basis integrals



  - known for any kinematic configuration
- Higher-loop: large set of integrals in the decomposition
  - they are not all independent
  - Direct integration is not possible

Master integrals for μe–scattering

- Four-point topologies for μe–scattering at two loops:



- Two loop integrals depending on two dimensionless ratios, s/m^2 , t/m^2
- Master integrals relevant for other 2→2 processes:
  - BhaBha scattering in QED [Gehrmann Remiddi 01, Bonciani Mastrolia Remiddi 04, …]
  - $t\bar{t}$ production in QCD [Bonciani, Ferroglia 08, Asatrian, Greub, Pecjak 08, …]
  - heavy-to-light quark decay in QCD [Bonciani, Ferroglia, Gehrmann 08, …]

Differential equations method

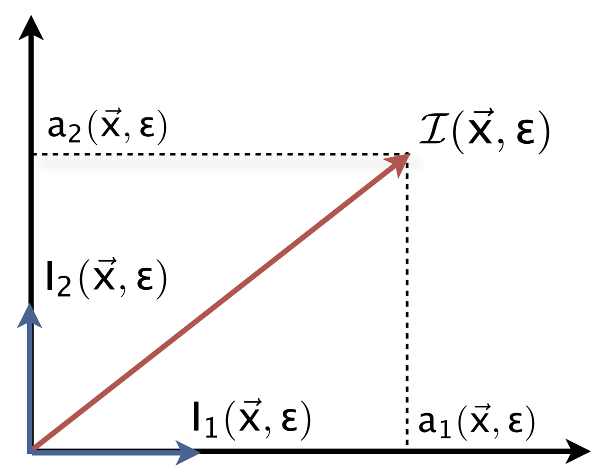
- Dimensionally-regularised Feynman integrals obey integration-by-parts ids

$$\int \prod_{j=1}^{\ell} \frac{d^d q_j}{(2\pi)^2} \frac{\partial}{\partial q_i^\mu} \left(v^\mu \frac{S_1^{b_1} \cdots S_r^{b_r}}{D_1^{a_1} \cdots D_n^{a_n}} \right) = 0 \qquad v^\mu \in \{q_i^\mu, p_i^\mu\}$$

[Chetyrkin, Tkachov 81]

  - Derive a basis of the space of Feynman integrals $\mathcal{I}(\vec{x}, \varepsilon)$

$$\mathcal{I}(\vec{x}, \varepsilon) = \sum_{i=1}^{N} a_i(\vec{x}, \varepsilon) I_i(\vec{x}, \varepsilon)$$



- Master integrals $\vec{I} = (I_1, I_2, \ldots, I_N)$ fulfil coupled 1st order PDEs in the kinematics

$$\frac{\partial}{\partial x_i} \vec{I}(\vec{x}, \varepsilon) = \mathbf{A}_i(\vec{x}, \varepsilon) \vec{I}(\vec{x}, \varepsilon)$$

[Kotikov 91, Remiddi 97, Gehrmann, Remiddi 00, …]

  - Computation of the master integrals: solve PDEs + boundary conditions

Differential equations method

- Coupled systems of PDEs from integration by parts

$$\frac{\partial}{\partial x_i}\vec{I}(\vec{x},\varepsilon) = \mathbf{A}_i(\vec{x},\varepsilon)\vec{I}(\vec{x},\varepsilon)$$

  - $\mathbf{A}_i(\vec{x},\varepsilon)$ are block-triangular
  - $\mathbf{A}_i(\vec{x},\varepsilon)$ are rational in $\vec{x}$ and ε

$$\mathbf{A}_i(\vec{x},\varepsilon) = \begin{pmatrix} \blacksquare & & & & & \\ * & \blacksquare & & & & \\ * & * & \blacksquare & \blacksquare & & \\ * & * & \blacksquare & \blacksquare & & \\ * & * & * & * & \blacksquare & \\ * & * & * & * & * & \blacksquare \end{pmatrix}$$

- Master integrals determined by series expansion for $\varepsilon \approx 0$

$$\vec{I}(\vec{x},\varepsilon) = \sum_{k=0}^{\infty} I^{(k)}(\vec{x})\varepsilon^k$$

  - PDEs for Taylor coefficients $\vec{I}^{(k)}(\vec{x})$ (mostly) triangularised
  - Solve the systems bottom-up

Canonical differential equations

- Systems of PDEs are not unique
  - Change of variables: $\vec{x} \to \vec{y}(\vec{x})$ $$\frac{\partial}{\partial y_i}\vec{I} = \left[\frac{\partial x_j}{\partial y_i}\mathbf{A}_j(\vec{y},\varepsilon)\right]\vec{I}$$
  - Change of basis: $\vec{I}(\vec{x},\varepsilon) = \mathbf{B}(\vec{x},\varepsilon)\vec{J}(\vec{x},\varepsilon)$ $$\frac{\partial}{\partial x_i}\vec{J} = \mathbf{B}^{-1}\left[\mathbf{A}_i\mathbf{B} - \frac{\partial}{\partial x_i}\mathbf{B}\right]\vec{J}$$
- Cast PDEs to canonical (=simplest) form

$$d\vec{I}(\vec{x},\varepsilon) = \varepsilon\left[\sum_{i=1}^{m}\mathbf{M}_i\,d\log\eta_i(\vec{x})\right]\vec{I}(\vec{x},\varepsilon)$$ [Henn 13]

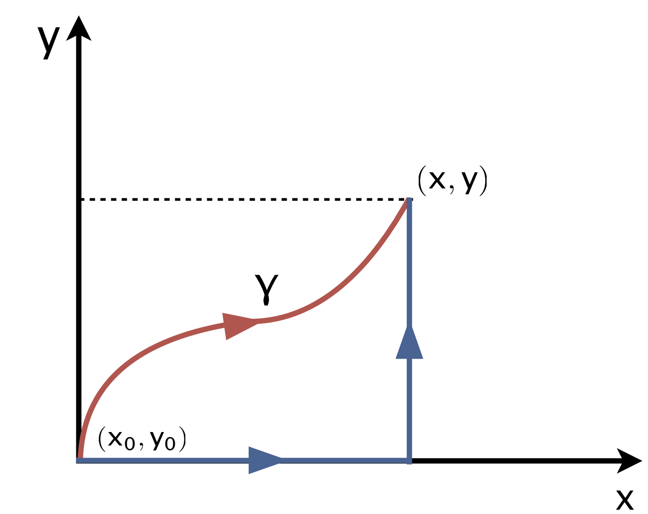
  - Order-by-order decoupling

$$d\vec{I}^{(k)}(\vec{x}) = \sum_{i=1}^{m} d\log\eta_i(\vec{x})\,\vec{I}^{(k-1)}(\vec{x})$$

  - Known classes of iterated integrals

Canonical differential equations

- Algorithmic solution in canonical form



$$\vec{I}(\vec{x},\epsilon) = \left[1 + \sum_{k=1}^{\infty} \int_{\gamma} d\mathbf{A} \ldots d\mathbf{A}\right] \vec{I}(\vec{x}_0,\epsilon)$$

  - Algebraic $\eta_i(\vec{x})$: Chen iterated integrals [Chen 77]

  - Rational $\eta_i(\vec{x})$: Generalised polylogarithms (GPLs)

$$A_i(\vec{x}) = \sum_{j=1}^{m} \frac{\mathbf{M}_j}{x_i - \omega_j}$$

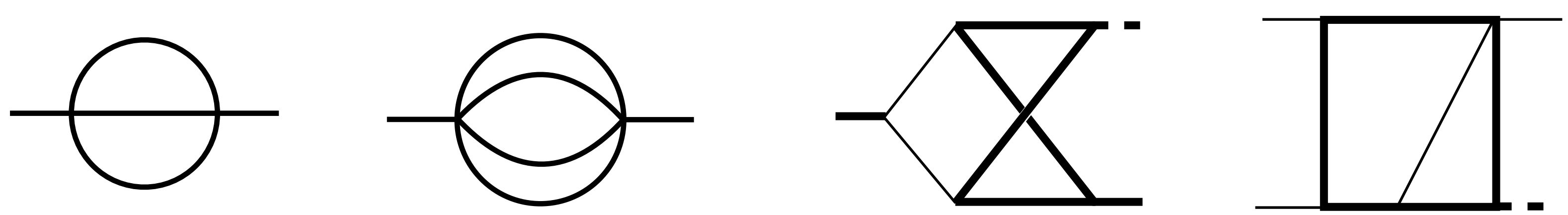
$$G(\vec{0}_n; x) = \frac{1}{n!} \mathrm{dlog}^n x \qquad G(\vec{\omega}_n; x) = \int_0^x \frac{dt}{t-\omega_1} G(\vec{\omega}_{n-1}; t)$$

[Goncharov 98, Remiddi, Vermaseren 99, Gehrmann, Remiddi 00, ...]

- Canonical form: easy to solve but hard to find

Finding the canonical form

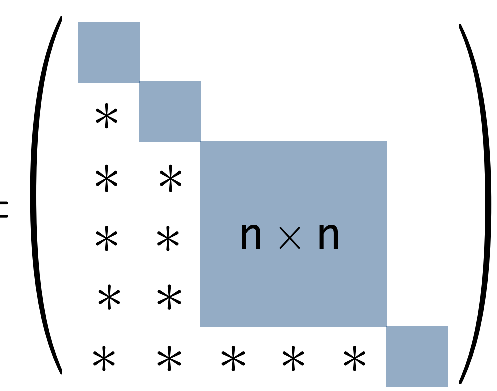
- Different strategies available
  - Unit leading singularity [Henn 13]
  - Magnus exponential [Argeri, Di Vita, Mastrolia et al 14]
  - Rational Ansätze for basis change [Gehrmann, Von Manteuffel, Tancredi et al 14,...]
  - Reduction to fuchsian form and eigenvalue normalisation [Lee 15]
  - Factorisation Picard–Fuchs operator [Adams, Chaubery, Wainzierl 17]
- But canonical forms might not exist



  - Integrals are not expressible in terms of GPLs
  - PDEs remain coupled even at $\varepsilon = 0$

$$\frac{\partial \vec{I}}{\partial x_i} = \mathbf{A}_{i,n\times n}(\vec{x},0)\vec{I} + \vec{N}(\vec{x},0)$$

$\mathbf{A}_{i,0} =$



Maximal–cuts

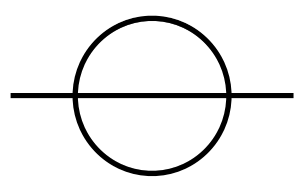
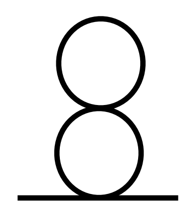
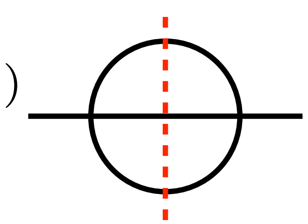
- Master integrals obey higher-order inhomogeneous PDEs

$$L^{(n)}_{x_i}\, I_1(\vec{x},0) = N_{1,i}(\vec{x}) \qquad L^{(n)}_{x_i} = p_0(\vec{x}) + \sum_{i=k}^{n} p_k(\vec{x})\,\frac{\partial^k}{\partial x_i^k}$$

  - Solve by integrated integration over the homogenous solutions

$$L^{(n)}_{x_i}\, h_{1k}(\vec{x}) = 0 \quad k = 1,\ldots,n$$

- Maximal-cuts solve the homogeneous PDEs

$L^{(2)}_{x}$  $= n(x)$  $\qquad L^{(2)}_{x}$  $= 0$ [Laporta, Remiddi 04]

  - Find n independent solutions

[**AP,** Tancredi 16–17,
Bosma, Sogaard, Zhang,17,
Harvey, Moriello, Schabinger 17,...]

$$\frac{\partial}{\partial x_i}\mathbf{H}(\vec{x}) = \mathbf{A}_{i,n\times n}(\vec{x},0)\mathbf{H}(\vec{x})$$

$$\mathbf{H}(\vec{x}) = \begin{pmatrix} \mathrm{MCut}_{\mathcal{C}_1}[I_1] & \mathrm{MCut}_{\mathcal{C}_2}[I_1] & \ldots & \mathrm{MCut}_{\mathcal{C}}[I_1] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{MCut}_{\mathcal{C}_1}[I_n] & \ldots & \ldots & \mathrm{MCut}_{\mathcal{C}_n}[I_n] \end{pmatrix}$$

Magnus exponential

- ε–linear PDEs in $\vec{x} = (s/m^2, t/m^2)$

$$\frac{\partial}{\partial x_i}\vec{I} = \left[\mathbf{A}_i^{(0)}(\vec{x}) + \varepsilon\,\mathbf{A}_i^{(1)}(\vec{x})\right]\vec{I} \qquad i = 1, 2$$

  - Solve PDEs for $\varepsilon = 0$

$$\frac{\partial}{\partial x_i}\mathbf{B}(\vec{x}) = \mathbf{A}_i^{(0)}\mathbf{B}(\vec{x})$$

  - Change of basis $\vec{I} = \mathbf{B}\vec{J}$

$$\frac{\partial}{\partial x_i}\vec{J} = \varepsilon\left[\mathbf{B}^{-1}\mathbf{A}_i^{(1)}\mathbf{B}\right]\vec{J}$$

- Formal solution: Magnus exponential

$$\mathbf{B}(\vec{x}) = \exp\left(\sum_k \Omega_k[\hat{A}_2^{(0)}](\vec{x})\right).\exp\left(\sum_j \Omega_j[A_1^{(0)}](\vec{x})\right)$$

$$\Omega_k[\mathbf{A}](t) = \begin{cases} \Omega_1[\mathbf{A}](t) = & \int dt_1 \mathbf{A}(t_1) \\ \Omega_2[\mathbf{A}](t) = & \int dt_1 dt_2 [\mathbf{A}(t_1), \mathbf{A}(t_2)] \\ \Omega_3[\mathbf{A}](t) = & \int dt_1 dt_2 dt_3 [\mathbf{A}(t_1), [\mathbf{A}(t_2), \mathbf{A}(t_3)]]_{(1,3)} \\ & \dots \end{cases}$$

[Magnus 54]

Canonical form for μe

- Magnus exponential:

$$\mathbf{B}(\vec{x}) = \exp\left(\sum_k \Omega_k[\hat{A}_2^{(0)}](\vec{x})\right).\exp\left(\sum_j \Omega_j[A_1^{(0)}](\vec{x})\right)$$

  - μe scattering: $\Omega_k[\mathbf{A}_i^{(0)}](\vec{x}) = 0$ for $k > 2$
  - change of basis: $\vec{I} = \mathbf{B}\vec{J}$
  - change of variables: $s = -m^2\, x \qquad t = -m^2\, \frac{(1-y)^2}{y}$

- Canonical form:

$$d\vec{J}(x,y,\varepsilon) = \varepsilon \left[\sum_{i=1}^{9} \mathbf{M}_i \, \mathrm{dlog}\, \eta_i(x,y)\right] \vec{J}(x,y,\varepsilon)$$

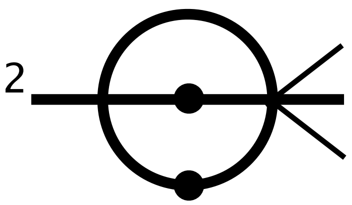
$\eta_1 = x$	$\eta_4 = y$	$\eta_7 = x + y$
$\eta_2 = 1 + x$	$\eta_5 = 1 + y$	$\eta_8 = 1 + xy$
$\eta_3 = 1 - x$	$\eta_6 = 1 - y$	$\eta_9 = 1 - y(1 - x - y)$

  - Solution in terms of GPLs in Euclidean region x>0, 0<y<1

[Mastrolia, Passera, **AP**, Schubert 17]

Boundary conditions

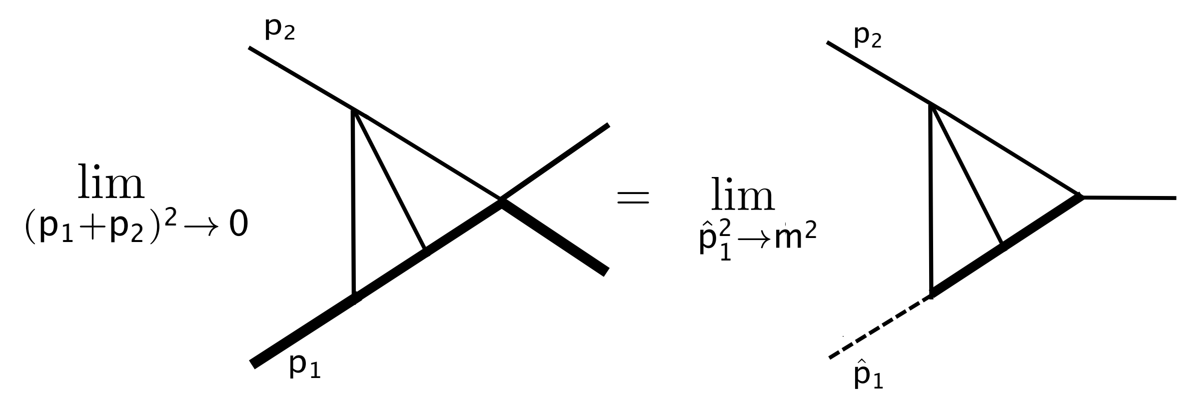
- Boundary conditions from physical information
  - external input:

$m^2\epsilon^2$  $= -\frac{1}{2}\zeta_2\epsilon^2 + \frac{1}{4}\left(12\zeta_2\log 2 - 7\zeta_3\right)\epsilon^3 + O(\epsilon^4)$

  - regularity at pseudo-thresholds η_k of the PDEs

$$\lim_{\eta_k \to 0} \mathbf{M}_k \vec{\mathrm{I}}(\vec{x}, \epsilon) = 0$$

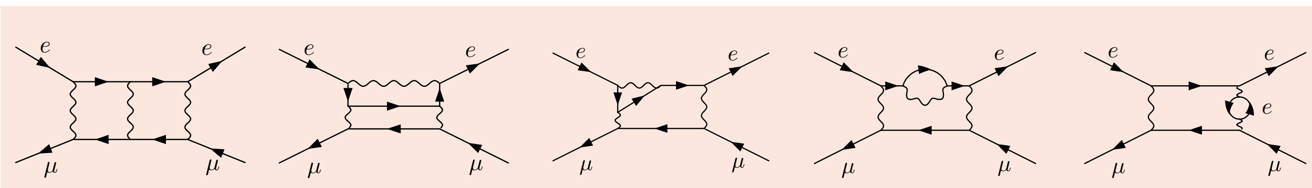
  - kinematic limits from auxiliary integrals

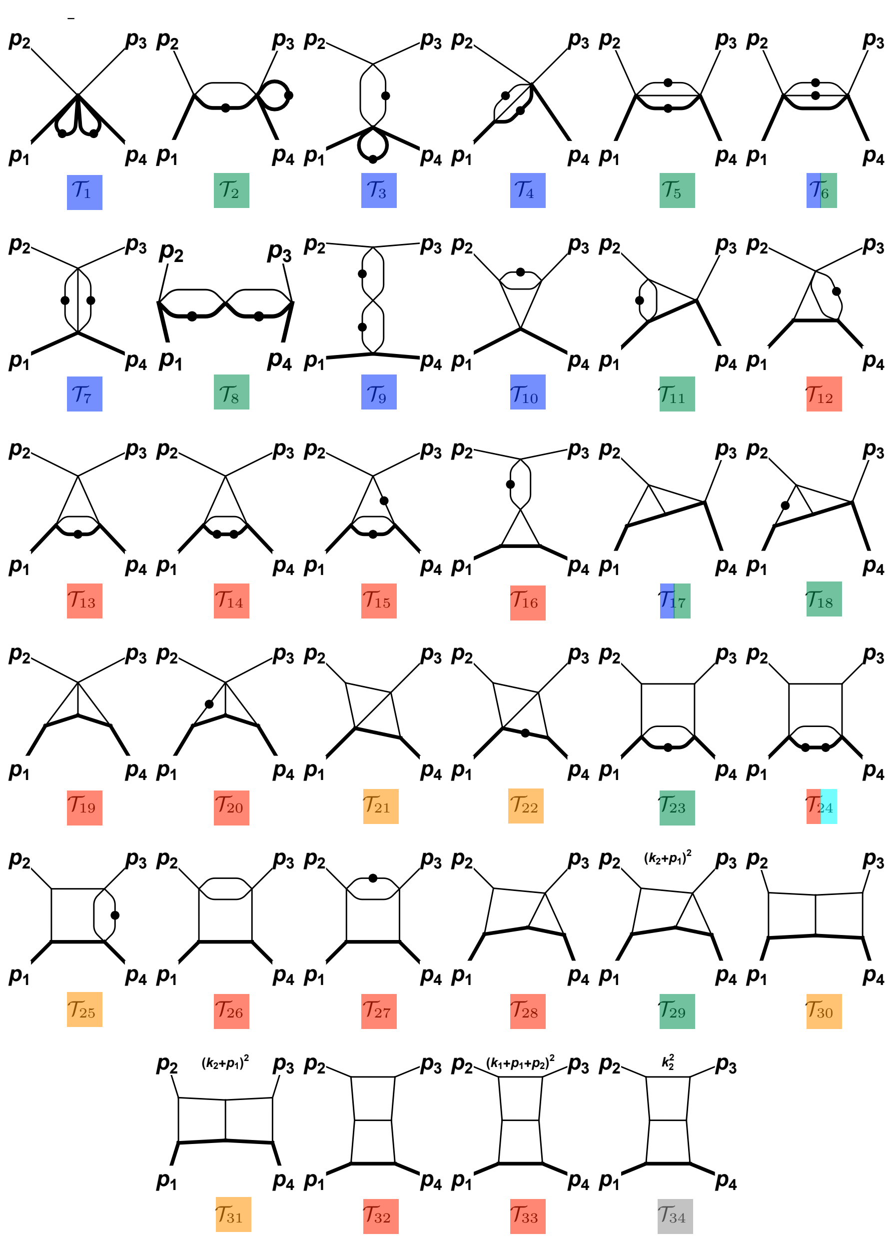


- Uniform combinations of constant GPLs fitted to ζ_k

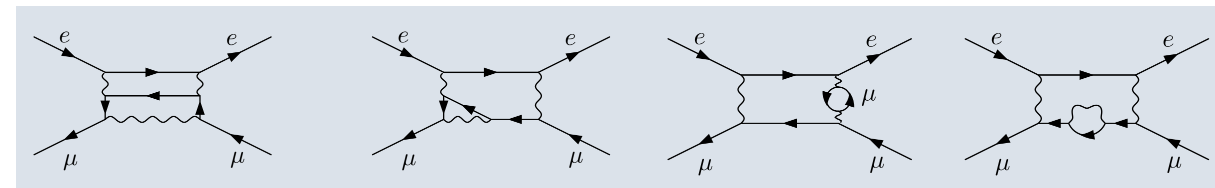
Planar integrals

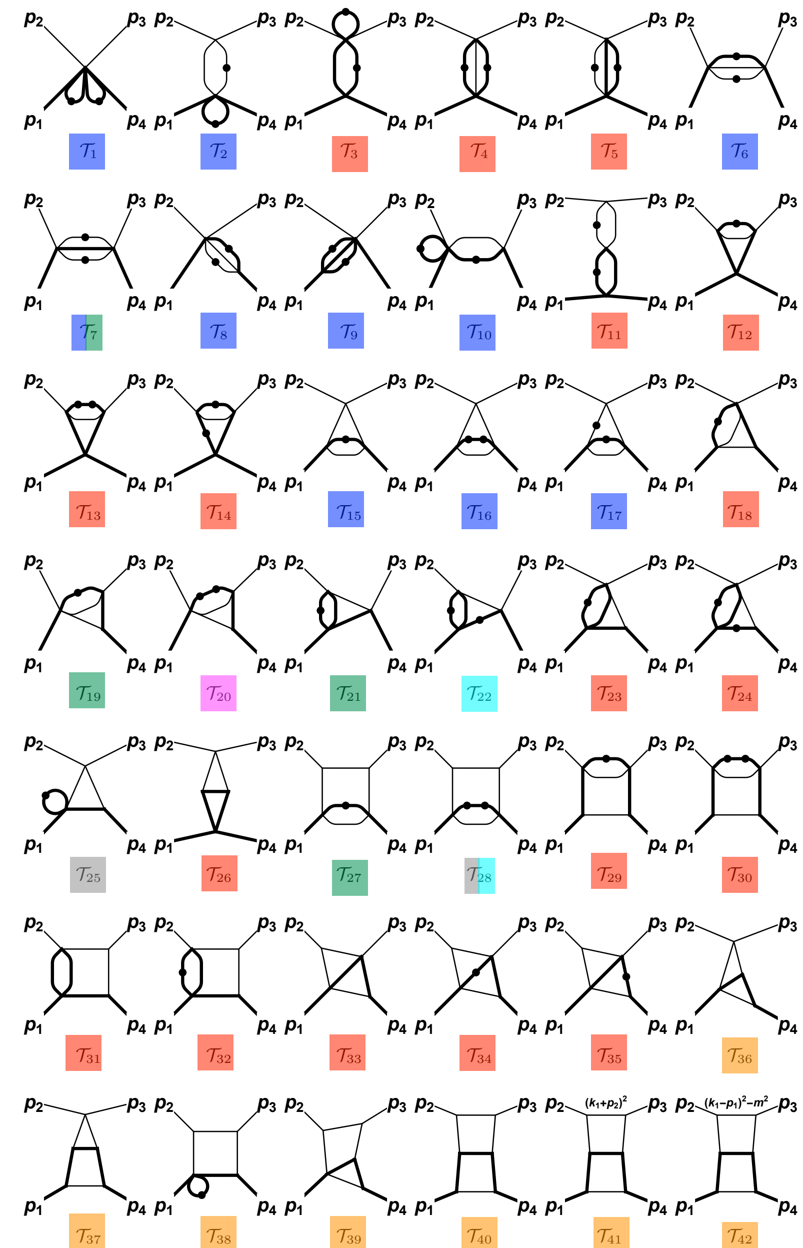
- 65 distinct master integrals evaluated





- Input
- $s \to 0$
- $t \to 4m^2$
- $u \to 2m^2$
- $s \to -m^2$
- $u \to \infty$



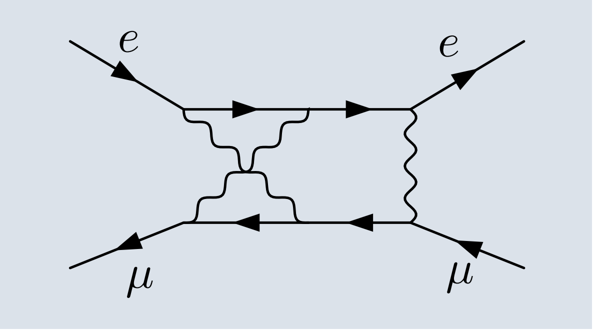


- Input
- $s \to 0$
- $t \to 0$
- $u \to m^2/2$
- $s \to -m^2$
- $t \to 4m^2$
- $s \to 2t - m^2 - \lambda_t$

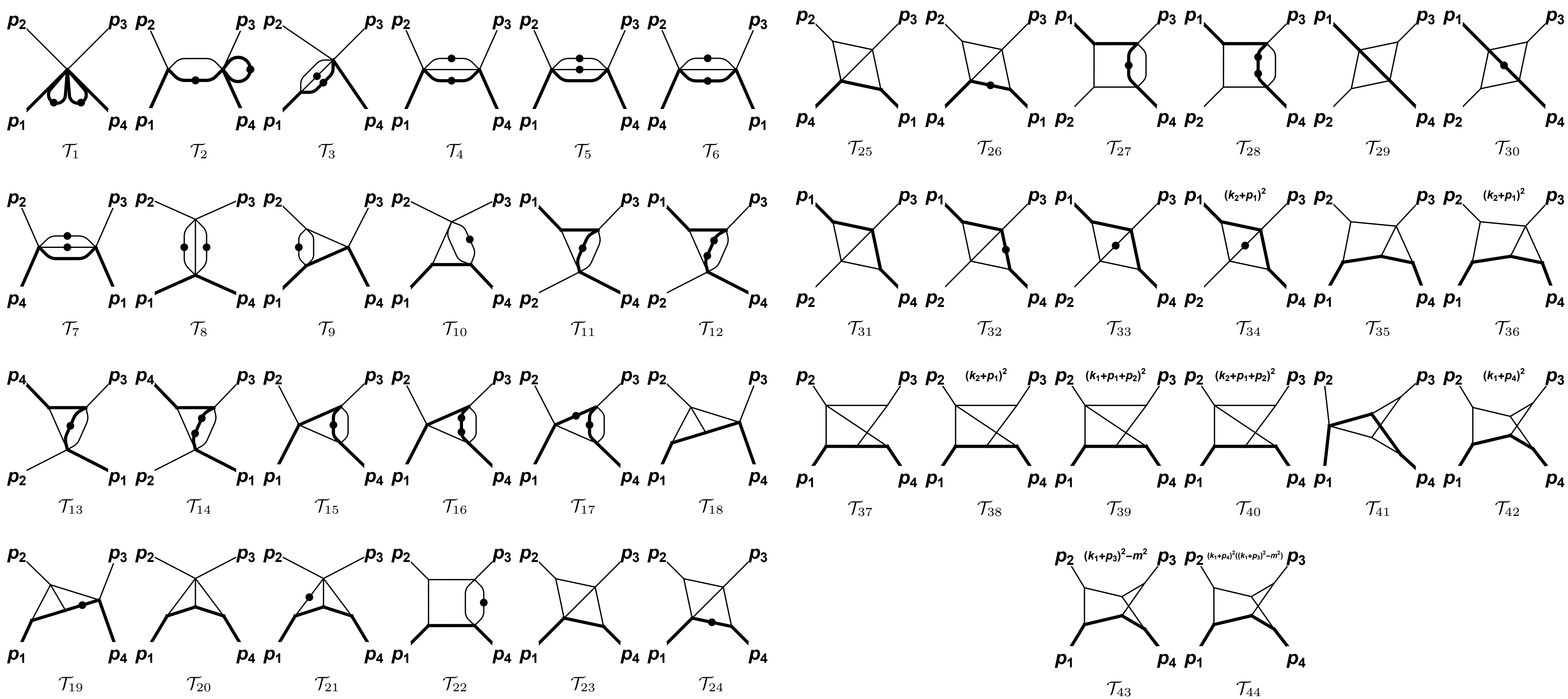
Non–planar integrals

- ε–linear PDEs in $\vec{x} = (s/m^2, t/m^2)$

$$\frac{\partial}{\partial x_i}\vec{I} = \left[\mathbf{A}_i^{(0)}(\vec{x}) + \varepsilon\,\mathbf{A}_i^{(1)}(\vec{x})\right]\vec{I} \qquad i = 1, 2$$



[Di Vita, Mastrolia, **AP**, Schubert xx]



Outlook and conclusions

- Experimental results for a_μ will improve soon
  - Theory prediction of a_μ^{Had} inadequate
  - Proposal for independent determination from μe scattering
- Unknown QED prediction at NNLO are required
  - Virtual amplitude decomposed to master integrals
  - All planar master integrals are now available in the $m_e = 0$ limit
  - The non-planar integrals will be completed (soon)
  - $|\mathcal{M}_\gamma^{(1)}|^2$ can be computed with available tools
- Lot of work towards a NNLO generator
  - First step: recover $m_e \neq 0$ effects

Outlook and conclusions

- Padova, 4th–5th September 2017

μe scattering:
Theory kickoff workshop

UNIVERSITÀ DEGLI STUDI DI PADOVA
INFN Istituto Nazionale di Fisica Nucleare

  - https://agenda.infn.it/conferenceDisplay.py?confId=13774

- Mainz, 19th–23th February 2018

The Evaluation of the Leading Hadronic Contribution to the Muon Anomalous Magnetic Moment

JG|U JOHANNES GUTENBERG UNIVERSITÄT MAINZ
mitp Mainz Institute for Theoretical Physics

  - https://indico.mitp.uni-mainz.de/event/128/

- Next: **Zürich** February 2019 (ask Adrian and Yannick!)