Local Integrands for Two-Loop QCD Amplitudes

based on [1511.06652], **[1606.02244]**, [1607.00311], [1706.09381]
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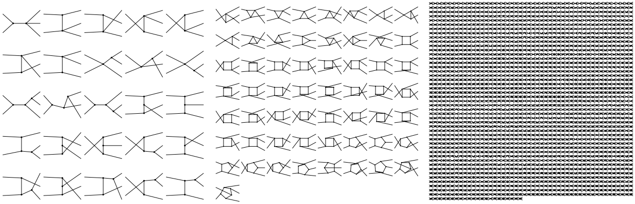
February 27, 2018

Complexity of Feynman diagrams

- Gauge-theory scattering amplitudes are traditionally formed as sums of gauge-dependent Feynman diagrams.
- Resulting expressions can be long and unmeaningful, e.g. $2 \to 3$ pure gluon scattering:



(a) LO (b) NLO (c) NNLO

- What "organizational principle" should we use to write loop integrands?
- Can we expose the physics without actually integrating?

Organizing D-dimensional loop integrands

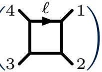
- Focus on gauge theories: typically use ordered color decomposition:

$$\mathcal{A}^{(L)}_{h_1\cdots h_n} = g^{n-2+2L} N_c^L \sum_{S_n} \mathrm{tr}(T^{a_1}\cdots T^{a_2}) A^{(L)}(1^{h_1},\cdots,n^{h_n}) + \mathcal{O}(N_c^{L-1})$$

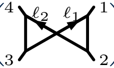
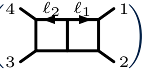
- T^a are SU(N_c) generators, g is the coupling, h_i are helicities.
- Ordered amplitudes given as sums of integrals of numerators Δ_T:

$$A^{(L)}(1,2,\cdots,n) = \frac{i}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\sum_T \mathcal{I}^D_T[\Delta_T]$$

- E.g.

$\mathcal{I}^D\Big($  $\Big)[\Delta] = \displaystyle\int \frac{d^D\ell}{(2\pi)^D}\frac{\Delta}{\ell^2(\ell-p_1)^2(\ell-p_{12})^2(\ell+p_4)^2}$

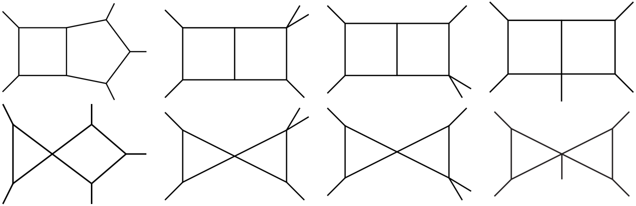
- Δ_T are not unique: we are free to move terms, e.g.

$\Delta\Big($  $\Big) = (\ell_1+\ell_2)^2\Delta\Big($  $\Big)$

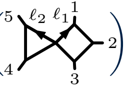
- Passarino-Veltman reduction: favor diagrams with fewer propagators.

Planar two-loop five-gluon all-plus integrand

[Badger, Frellesvig & Zhang '13]



- E.g. box-triangle numerator, $s_{ij} = (p_i + p_j)^2$, ω_{123} spurious direction.

$\Delta\Bigg($ 

$\Bigg) = -\frac{s_{12}\mathrm{tr}_+(1345)}{2s_{13}}(2\ell_1\cdot\omega_{123}+s_{23})\left(F_2+F_3\frac{(\ell_1+\ell_2)^2+s_{45}}{s_{45}}\right)$

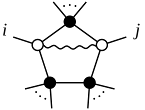
- s_{13} is an unphysical pole: doesn't appear in the integrated result [Gehrmann, Henn, Lo Presti '15]
- Other helicities @ 5 points, 2 loops computed numerically:
  - [Badger, Brønnum-Hansen, Hartanto & Peraro '17]
  - [Abreu, Cordero, Ita, Page & Zheng '17]

Local integrands in planar $\mathcal{N} = 4$ SYM

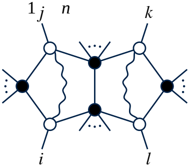
[Arkani-Hamed, Bourjaily, Cachazo & Trnka '12]

- Compact L-loop MHV integrands

$$A^{(1)}(1, 2, \cdots, n) = \sum_{i<j}$$



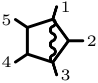
$$A^{(2)}(1, 2, \cdots, n) = \frac{1}{2} \sum_{i<j<k<l<i}$$



- Two-loop N^{k}MHV now computed [Bourjaily & Trnka '15]
- These expressions enjoy manifest physical properties at the integrand level: IR structure, locality, soft behavior, etc.
- BUT, they rely on the momentum twistor formalism [Hodges '09].
- This limits applicability to $D = 4$, color ordered amplitudes.

Moving to Dirac traces

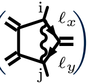
- Local integrands can be re-expressed as Dirac traces:

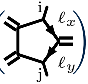


$$\propto \int \frac{d^4\ell}{(2\pi)^4} \frac{\mathrm{tr}_+(1(\ell - p_1)(\ell - p_{12})3)}{\ell^2(\ell - p_1)^2(\ell - p_{12})^2(\ell + p_{45})^2(\ell + p_5)^2}$$

where $\mathrm{tr}_\pm(ij\cdots k) = \frac{1}{2}\,\mathrm{tr}((1 \pm \gamma_5)\not{p}_i\not{p}_j\cdots\not{p}_k)$.

- Expressions work in D dimensions! Define regulated integrals as

$\mathcal{I}^D\Bigg($ 

$\Bigg)\,[\mathcal{P}(p_i,\ell)] \equiv \mathcal{I}^D\Bigg($ 

$\Bigg)\,[\mathrm{tr}_+\,(i\ell_x\ell_y j)\,\mathcal{P}(p_i,\ell)]$

- No reference to color ordering or on-shell conditions.
- Applicable to a much wider class of theories.

Generalizing to $D = 4 - 2\epsilon$

[Bern & Morgan '95]

- Generalize momenta to D dimensions by writing

$$\ell_i = (\bar{\ell}_i, \ell_i^{[-2\epsilon]}) \qquad \mu_{ij} = -\ell_i^{[-2\epsilon]} \cdot \ell_j^{[-2\epsilon]}$$
$$p_i = (p_i, 0) \qquad \mu^2 = -(\ell^{[-2\epsilon]})^2$$

- Dirac traces formally require a D-dimensional Clifford algebra:

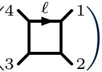
$$\{\bar{\gamma}_i, \gamma_5\} = 0 \qquad [\gamma_i^{[-2\epsilon]}, \gamma_5] = 0$$

where $\gamma_5 = i\bar{\gamma}_0\bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3$.

- D-dimensional traces decompose as

$$\begin{aligned}\mathrm{tr}_\pm(i_1 \cdots i_k \ell_x \ell_y i_{k+1} \cdots i_n) = {} & \mathrm{tr}_\pm(i_1 \cdots i_k \bar{\ell}_x \bar{\ell}_y i_{k+1} \cdots i_n) \\ & - \mu_{xy} \mathrm{tr}_\pm(i_1 \cdots i_n)\end{aligned}$$

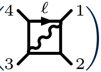
An invitation: the box integral in $D = 4 - 2\epsilon$

$\mathcal{I}^D \Bigg($  $\Bigg)$

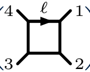
$$= \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{\ell^2(\ell - p_1)^2(\ell - p_{12})^2(\ell + p_4)^2}$$

$$= i\frac{c_\Gamma}{st}\left(\frac{2}{\epsilon^2}\left((-s)^{-\epsilon} + (-t)^{-\epsilon}\right) - \ln^2\left(\frac{s}{t}\right) - \pi^2\right) + \mathcal{O}(\epsilon)$$

- $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$, $c_\Gamma = (4\pi)^{-D/2}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$.
- IR divergences occur between pairs of massless external legs,
  - soft, e.g. $\ell \to 0$;
  - collinear, e.g. $\ell // p_1$.
- Regulate IR using local box numerator:

$\mathcal{I}^D \Bigg($  $\Bigg)$

$$= \int \frac{d^D \ell}{(2\pi)^D} \frac{\mathrm{tr}_+(1(\ell - p_1)(\ell - p_{12})3)}{\ell^2(\ell - p_1)^2(\ell - p_{12})^2(\ell + p_4)^2}$$

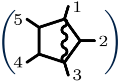
$= (-1 + 2\epsilon)u(4\pi)^2\mathcal{I}^{D+2} \Bigg($  $\Bigg)$

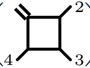
$$= -i\frac{c_\Gamma}{2}\left(\ln^2\left(\frac{s}{t}\right) + \pi^2\right) + \mathcal{O}(\epsilon)$$

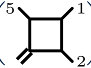
Five-point $\mathcal{N} = 4$ local integrands

- Armed with this notation, we can write $\mathcal{N} = 4$ MHV integrands as

$$A^{(1)}_{\text{MHV}}(1,2,3,4,5) = \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \times$$

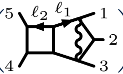
$$\Bigg(\frac{\mathrm{tr}_+(1345)}{s_{13}} \mathcal{I}\Big($$


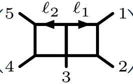
$$\Big) - s_{23}s_{34}\mathcal{I}\Big($$


$$\Big) - s_{12}s_{15}\mathcal{I}\Big($$


$$\Big)\Bigg)$$

$$A^{(2)}_{\text{MHV}}(1,2,3,4,5) = \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \times$$

$$\sum_{\text{cyclic}} \Bigg(\frac{s_{45}\,\mathrm{tr}_+(1345)}{s_{13}} \mathcal{I}\Big($$


$$\Big) + s_{12}s_{45}s_{15}\mathcal{I}\Big($$


$$\Big)\Bigg)$$

- $s_{ij} = (p_i + p_j)^2$
- All expressions are free of spurious singularities because

$$\frac{\mathrm{tr}_+(1345)}{s_{13}}\,\mathrm{tr}_+(1(\ell_1 - p_1)(\ell_1 - p_{12})3) = \mathrm{tr}_+(1(\ell_1 - p_1)(\ell_1 - p_{12})345)$$

All-plus helicity amplitudes

[Bern, Dixon, Dunbar & Kosower '96]

- Tree amplitudes vanish:

$$A^{(0)}(1^+2^+\cdots n^+) = 0$$

- One-loop amplitudes integrate to rational functions:

$$A^{(1)}(1^+2^+\cdots n^+) = -\frac{i}{48\pi^2}\sum_{1\le i_1<i_2<i_3<i_4\le n}\frac{\mathrm{tr}_-(i_1 i_2 i_3 i_4)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle} + \mathcal{O}(\epsilon)$$

- Also satisfy dimension-shifting relation with $\mathcal{N}=4$ MHV:

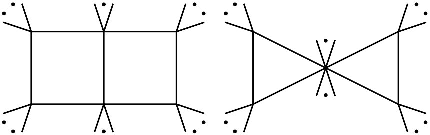
$$A^{(1)}(1^+2^+\cdots n^+) = -2\epsilon(1-\epsilon)(4\pi)^2\left.\frac{A^{(1),[\mathcal{N}=4]}(1^-2^-3^+\ldots n^+)}{\langle 12\rangle^4}\right|_{D\to D+4}$$

- Corresponds to numerator correspondence:

$$\Delta_T(1^+2^+\cdots n^+;\ell) = \frac{(D_s-2)\mu^4}{\langle 12\rangle^4}\Delta_T^{[\mathcal{N}=4]}(1^-2^-3^+\cdots n^+;\ell)$$

Two-loop correspondence

- At 2 loops we distinguish between genuine two-loop (left) and one-loop-squared (right) topologies:



- Genuine two-loop numerators related to $\mathcal{N} = 4$:

$$\Delta_T(1^+2^+\cdots n^+;\ell_1,\ell_2) = \frac{F_1}{\langle 12\rangle^4}\Delta_T^{[\mathcal{N}=4]}(1^-2^-3^+\cdots n^+;\ell_1,\ell_2)$$

$$F_1 = (D_s-2)(\mu_{11}\mu_{22}+(\mu_{11}+\mu_{22})^2+2\mu_{12}(\mu_{11}+\mu_{22}))+16(\mu_{12}^2-\mu_{11}\mu_{22})$$

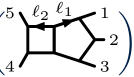
- One-loop squared split into terms proportional to (D_s-2) and $(D_s-2)^2$,

$$F_2 = 4(D_s-2)\mu_{12}(\mu_{11}+\mu_{22})$$
$$F_3 = (D_s-2)^2\mu_{11}\mu_{22}$$

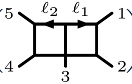
- F_2 vanishes on integration; F_3 gives rise to rational terms.

The two-loop five-gluon all-plus integrand

- Genuine two-loop numerators follow $\mathcal{N}=4$:

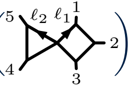
$\Delta\Big($  $\Big)$

$$= s_{45}\,\mathrm{tr}_+(1(\ell_1-p_1)(\ell_1-p_{12})345)F_1$$

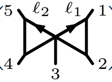
$\Delta\Big($  $\Big)$

$$= -s_{12}s_{45}s_{15}F_1$$

- One-loop-squared numerators have no $\mathcal{N}=4$ counterparts:

$\Delta\Big($  $\Big)$

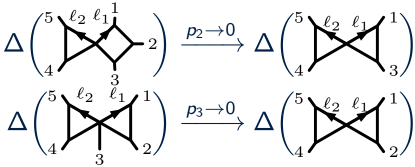
$$= \mathrm{tr}_+(1(\ell_1-p_1)(\ell_1-p_{12})345)\times \left(F_2+F_3\frac{s_{45}+(\ell_1+\ell_2)^2}{s_{45}}\right)$$

$\Delta\Big($  $\Big)$

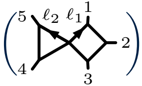
$$= \mathrm{tr}_+(1245)(F_2+F_3)+\frac{F_3}{s_{12}s_{45}}\Big(\mathrm{tr}_+(123\ell_1\ell_2 345) + (s_{12}s_{45}s_{15}+(s_{12}+s_{45})\,\mathrm{tr}_+(1245))(\ell_1+\ell_2)^2\Big)$$

Soft limits on external legs

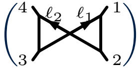
- Numerators obey soft limits on external legs:



- E.g. consider

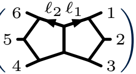
Δ  $= \mathrm{tr}_+(1(\ell_1 - p_1)(\ell_1 - p_{12})345)\left(F_2 + F_3 \frac{s_{45} + (\ell_1+\ell_2)^2}{s_{45}}\right)$

- Reproduces four-point result [Bern, Dixon & Kosower '00]:

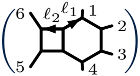
Δ  $= F_2 + F_3\left(\frac{s + (\ell_1+\ell_2)^2}{s}\right)$

The six-gluon all-plus integrand

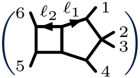
- Genuine two-loop topologies remarkably compact:

Δ 

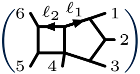
$$= s_{123}\,\mathrm{tr}_+(1(\ell_1-p_1)(\ell_1-p_{12})34(\ell_2-p_{56})(\ell_2-p_6)6)F_1$$

Δ 

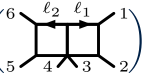
$$= -s_{56}\,\mathrm{tr}_+(123456)\mu_{11}F_1$$

Δ 

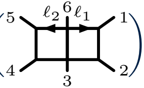
$$= s_{56}\,\mathrm{tr}_+(1(\ell_1-p_1)(\ell_1-p_{123})456)F_1$$

Δ 

$$= s_{56}\,\mathrm{tr}_+(1(\ell_1-p_1)(\ell_1-p_{12})3(4+5)6)F_1$$

Δ 

$$= -s_{12}s_{56}s_{61}F_1$$

Δ 

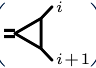
$$= -s_{12}s_{45}s_{234}F_1$$

- We also have expressions for one-loop-squared, but less compact.

Universal infrared structure

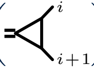
[Catani, Dittmaier & Trócsányi '01]

- Tree-level all-plus amplitudes are vanishing, so two-loop IR divergences are proportional to one-loop amplitudes:

$$A^{(2)}(1^+2^+\cdots n^+) = i\sum_{i=1}^{n} s_{i,i+1}\mathcal{I}^D\Bigg($$


$$\Bigg)A^{(1)}(1^+2^+\cdots n^+) + \mathcal{O}(\epsilon^0)$$

- Triangle integral carries IR poles:

$$\mathcal{I}^D\Bigg($$


$$\Bigg) = -i\frac{c_\Gamma}{\epsilon^2}(-s_{i,i+1})^{-1-\epsilon}$$

- Reproducing this behavior requires us to find the IR divergences up to $\mathcal{O}(\epsilon^{-1})$ of all two-loop integrals.
- Two-loop integrals are broken into sums of regions with soft singularities and evaluated in their respective limits.

Soft limits of two-loop integrals

- Soft and collinear regions involve taking $\ell_i^{[-2\epsilon]} \to 0$.
- All two-loop integrals contain

$$F_1 = (D_s - 2)(\mu_{11}\mu_{22} + (\mu_{11}+\mu_{22})^2 + 2\mu_{12}(\mu_{11}+\mu_{22})) + 16(\mu_{12}^2 - \mu_{11}\mu_{22})$$
$$F_2 = 4(D_s - 2)\mu_{12}(\mu_{11} + \mu_{12})$$
$$F_3 = (D_s - 2)^2 \mu_{11}\mu_{22}$$

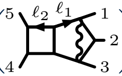
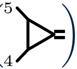
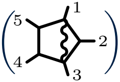
- In these limits,

$$F_1 \xrightarrow{\ell_1^{[-2\epsilon]} \to 0} (D_s - 2)\mu_{22}^2 \qquad F_1 \xrightarrow{\ell_2^{[-2\epsilon]} \to 0} (D_s - 2)\mu_{11}^2$$
$$F_2 \xrightarrow{\ell_i^{[-2\epsilon]} \to 0} 0 \qquad F_3 \xrightarrow{\ell_i^{[-2\epsilon]} \to 0} 0$$

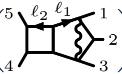
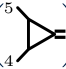
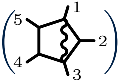
- No divergences beyond $\mathcal{O}(\epsilon^{-2})$ as only one of the loop momenta can enter a soft or collinear region at a time.

Soft limit of the pentabox

- The regulated pentabox has only one soft region:

$$\mathcal{I}^D\left(\vphantom{\int}\right.$$  $$\left.\vphantom{\int}\right)[F_1] \xrightarrow{\ell_2 \to p_5} (D_s - 2)\mathcal{I}^D\left(\vphantom{\int}\right.$$  $$\left.\vphantom{\int}\right)\mathcal{I}^D\left(\vphantom{\int}\right.$$  $$\left.\vphantom{\int}\right)[\mu^4].$$

- The other two supposedly soft limits $\ell_1 \to p_1$ and $\ell_1 \to p_{12}$ are finite because $\mathrm{tr}_+(1(\ell_1 - p_1)(\ell_1 - p_{12})3) \to 0$. So,

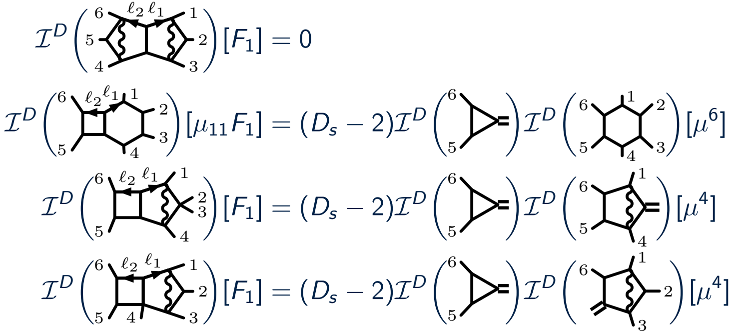
$$\mathcal{I}^D\left(\vphantom{\int}\right.$$  $$\left.\vphantom{\int}\right)[F_1] = (D_s - 2)\mathcal{I}^D\left(\vphantom{\int}\right.$$  $$\left.\vphantom{\int}\right)\mathcal{I}^D\left(\vphantom{\int}\right.$$  $$\left.\vphantom{\int}\right)[\mu^4]$$

$$+ \mathcal{O}(\epsilon^0).$$

- If we weren't using local integrands, this would only be true up to $\mathcal{O}(\epsilon)$.

Six-point integrals

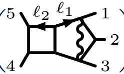
- Up to terms $\mathcal{O}(\epsilon^0)$,

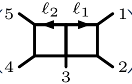


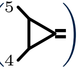
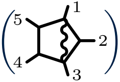
- IR divergences always come from boxes: these split off to form triangle integrals.

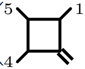
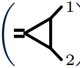
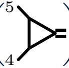
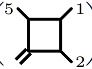
Five-gluon infrared decomposition (1)

$\langle 12\rangle \cdots \langle 51\rangle A^{(2)}(1^+,2^+,3^+,4^+,5^+)$

$= i \sum_{\text{cyclic}} \Bigg(\dfrac{s_{45}\,\mathrm{tr}_+(1345)}{s_{13}} \mathcal{I}^D \Bigg($  $\Bigg) [F_1]$

$- s_{12} s_{45} s_{15} \mathcal{I}^D \Bigg($  $\Bigg) [F_1] + \cdots \Bigg)$

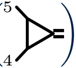
$= i(D_s - 2) \sum_{\text{cyclic}} \Bigg(\dfrac{s_{45}\,\mathrm{tr}_+(1345)}{s_{13}} \mathcal{I}^D \Bigg($  $\Bigg) \mathcal{I}^D \Bigg($  $\Bigg) [\mu^4]$

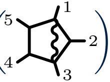
$- s_{12} s_{45} s_{15} \Bigg(\mathcal{I}^D \Bigg($  $\Bigg) [\mu^4] \mathcal{I}^D \Bigg($  $\Bigg) + \mathcal{I}^D \Bigg($  $\Bigg) \mathcal{I}^D \Bigg($  $\Bigg) \Bigg) \Bigg)$

$+ \mathcal{O}(\epsilon^0)$

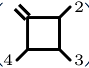
Five-gluon infrared decomposition (2)

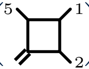
$$\langle 12\rangle\cdots\langle 51\rangle A^{(2)}(1^+,2^+,3^+,4^+,5^+)$$

$$= i(D_s-2)\sum_{\text{cyclic}} s_{45}\mathcal{I}^D\Bigg($$


$$\Bigg)\Bigg(\frac{\mathrm{tr}_+(1345)}{s_{13}}\mathcal{I}^D\Bigg($$


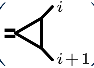
$$\Bigg)[\mu^4]$$

$$- s_{23}s_{34}\mathcal{I}^D\Bigg($$


$$\Bigg)[\mu^4] - s_{12}s_{15}\mathcal{I}^D\Bigg($$


$$\Bigg)[\mu^4]\Bigg) + \mathcal{O}(\epsilon^0)$$

- Therefore,

$$A^{(2)}(1^+2^+3^+4^+5^+) = i\sum_{i=1}^{n} s_{i,i+1}\mathcal{I}^D\Bigg($$


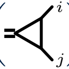
$$\Bigg)A^{(1)}(1^+2^+3^+4^+5^+) + \mathcal{O}(\epsilon^0)$$

as expected.

- The procedure works equally well for six-gluon scattering.

Subleading color structure

- Until now we have focused on the planar limit. Disposing of momentum twistors makes it possible to consider nonplanar topologies.

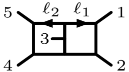
$$\mathcal{A}^{(2)}_{++\cdots+} = i \sum_{i \neq j} s_{ij}\, \mathcal{I}^D \Bigg($$


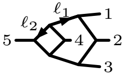
$$\Bigg) T_i \cdot T_j \circ \mathcal{A}^{(1)}_{++\cdots+} + \mathcal{O}(\epsilon^0)$$

  where for external gluons $T^a_{bc} = i\, f^{bac}$.
- Need a way to assign color factors to non-cubic diagrams:

$$\tilde{\Delta}_T(a_i, p_i, \ell_i) = \sum_\sigma c_T(a_i)\, \Delta_T(p_i, \ell_i)$$

- Also need to find local integrands for nonplanar topologies, e.g.





- Studies of subleading color have already been initiated in $\mathcal{N} = 4$ [Bern, Herrmann, Litsey, Stankowicz & Trnka '15].

Generalizing to alternate helicities

- For mixed external helicities we lose the $\mathcal{N}=4$ connection. So how to make use of local integrand structures?
- Return to an integrand-reduction based procedure.
- Integrand written as polynomial of irreducible scalar products (ISPs):

$$\Delta_T = \sum_{i_1,\ldots,i_n} c_{T;i_1 i_2\cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n},$$

- ISPs form basis of objects which cannot be expressed in terms of propagators. Generally satisfy polynomial relationships.
- Replace spurious components $\ell_i \cdot \omega$ in favor of local integrands, e.g. $\mathrm{tr}_+(1(\ell - p_1)(\ell - p_{12})3)$.
- Can we learn more from local integrands in the N^{k}MHV sectors of $\mathcal{N}=4$?

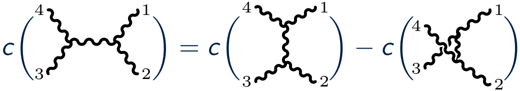
Color-kinematics duality

[Bern, Carrasco & Johansson '08], [Johansson & Ochirov '15]

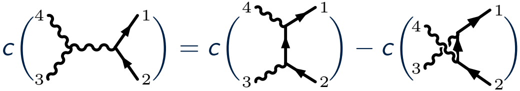
$$\mathcal{A}^{(L)} = \sum_{\text{cubic diagrams j}} \int \frac{d^{DL}\ell}{(2\pi)^{DL}} \frac{1}{S_j} \frac{c_j n_j(\ell)}{\mathcal{D}_j(\ell)}$$

- Color factors c_j satisfy Jacobi/commutation relations:

$$f^{a_1 a_2 b} f^{b a_3 a_4} = f^{a_4 a_1 b} f^{b a_2 a_3} - f^{a_2 a_4 b} f^{b a_3 a_1}$$



$$T^{b}_{i_1 \bar{\imath}_2} f^{b a_3 a_4} = T^{a_3}_{i_1 \bar{\jmath}} T^{a_4}_{j \bar{\imath}_2} - T^{a_4}_{i_1 \bar{\jmath}} T^{a_3}_{j \bar{\imath}_2} = [T^{a_3}, T^{a_4}]_{i_1 \bar{\imath}_2}$$



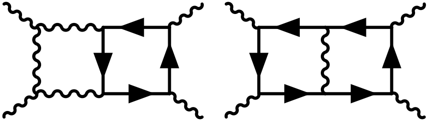
- Color-dual numerators n_T chosen to ensure that

$$c_i = c_j - c_k \iff n_i = n_j - n_k$$

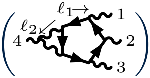
- Advantages: natural color basis, small number of planar masters.

Two-loop $\mathcal{N} = 2$ SQCD
[Johansson, Kälin & G.M. '17]

- At four points, only two masters required:



- Fermion lines are hypermultiplets, inherited from full $\mathcal{N} = 4$ spectrum. Numerators extend to arbitrary number of flavors.
- Simple resulting MHV expressions, e.g.

$n\Bigg($ 

$\Bigg)$ $= \dfrac{[12][34]}{\langle 12\rangle\langle 34\rangle} \times \dfrac{1}{15}\big(p_4 \cdot (\ell_1 - \ell_2) - 2\ell_1 \cdot \ell_2\big)$

- Can be extended to include massive matter.
- Ongoing $\mathcal{N} = 1$ SYM calculation, then hope to study $\mathcal{N} = 0$ and 3 loops.

Color-dual two-loop five-gluon all-plus

[G.M. & O'Connell '15]

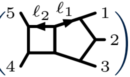
- Locality easily realized at four points using perm-invariant overall factor

$$\frac{[12][34]}{\langle 12\rangle\langle 34\rangle} = stA^{(0)}(1234)$$

- Unclear what generalization to use at higher points, try:

$$\beta_{12345} = \frac{[12][23][34][45][51]}{4\,\epsilon(1234)}, \quad \gamma_{12} = \beta_{12345} - \beta_{21345} = \frac{[12]^2[34][45][35]}{4\,\epsilon(1234)}$$

- Only need one master: the pentabox,

$n\Bigg($  $\Bigg) = \gamma_{12}\,m_1 + \gamma_{13}\,m_2 + \gamma_{14}\,m_3 + \gamma_{23}\,m_4 + \gamma_{24}\,m_5 + \gamma_{34}\,m_6$

- But, the resulting solution is large: 12 powers of loop momentum!
- We know the issue: can't find a local basis!
- Now would be a good time to look for local integrands!

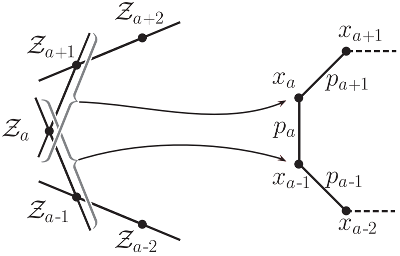
Conclusions

- D-dimensional local integrands are a powerful tool in the context of dimensionally-regulated amplitudes.
- All-plus Yang-Mills is a convenient testing ground: we now have local integrand representations of the five- and six-gluon amplitudes at two-loop order.
- The presentation highlights three important physical properties before integration:
  - Infrared structure;
  - Absence of spurious singularities (nonlocalities);
  - Manifest soft limits on external legs.
- Moving forward, we hope to explore both nonplanar amplitudes and mixed external helicities.
- An important next step is understanding the mechanism by which integrals factorize.

Thanks for listening!

Momentum twistor geometry
[Hodges '09]

- Momentum twistor geometry expressed using dual coordinates x_a:



- Momentum twistors $Z = (\lambda, \mu)$ defined in $\mathbb{CP}^3$ as

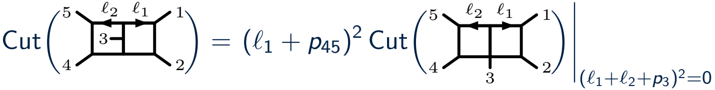
$$p_a = x_a - x_{a-1} \qquad\qquad \mu_{\dot{a}} = \lambda^a x_{a\dot{a}}$$

- Momentum twistors automatically imply
  - $\sum_i p_i = 0$ (momentum conservation)
  - $p_i^2 = 0$ (massless on-shell condition)
- But, we are limited to massless, color-ordered amplitudes in $D = 4$.

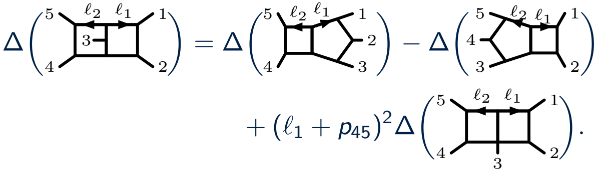
Nonplanar numerators from BCJ relations

[Badger, G.M., Ochirov & O'Connell '15]

- Maximal cuts of diagrams satisfy relations:



- This implies an on-shell relationship between numerators:



- This identity is a Jacobi identity: the last term is a correction.
- Expressions still need to be continued off-shell!

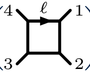
Integrand reduction via polynomial division

[Zhang '12] and others!

- Use diagrams with smallest number of propagators, e.g. @ 1 loop include only boxes, triangles, bubbles & tadpoles.
- Express Δ_T as polynomials of ISPs x_i:

$$\Delta_T = \sum_{i_1,\ldots,i_n} c_{T;i_1 i_2 \cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n},$$

- ISPs x_i cannot be expressed in terms of propagators of T. Could be
- $x_i = \mu_{ij} \equiv -\ell_i^{[-2\epsilon]} \cdot \ell_j^{[-2\epsilon]}, \ell_i \cdot \omega$, where $\omega \cdot p_i = 0$.
- Unique coefficients $c_{T;i_1 i_2 \cdots i_n}$ generally extracted from on-shell data.
- ISPs x_i satisfy higher-order relations, resulting from Gram matrices.

$\Delta\Bigg($ 

$\Bigg)$ $= c_0 + c_1 \ell \cdot \omega + c_2 \mu^2 + c_3 \mu^2 \ell \cdot \omega + c_4 \mu^4 + \cdots$

- Two powers of $\ell \cdot \omega$ do not appear because

$(\ell \cdot \omega)^2 - \mu^2 =$ linear combination of ℓ^2, $(\ell - p_1)^2$, $(\ell - p_{12})^2$, $(\ell + p_4)^2$

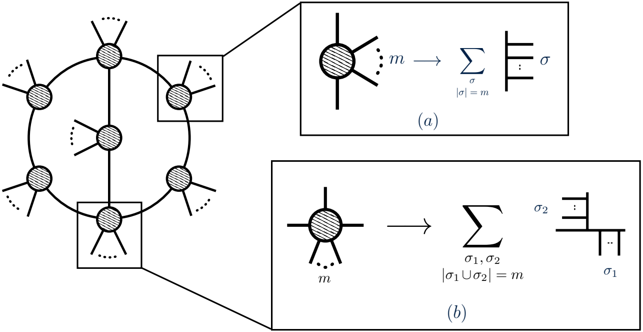
Multi-peripheral color decomposition

[Badger, G.M., Ochirov & O'Connell '15], [Ochirov & Page '16]

- Think about color-dressed unitarity cuts.
- Use DDM basis on tree amplitudes:

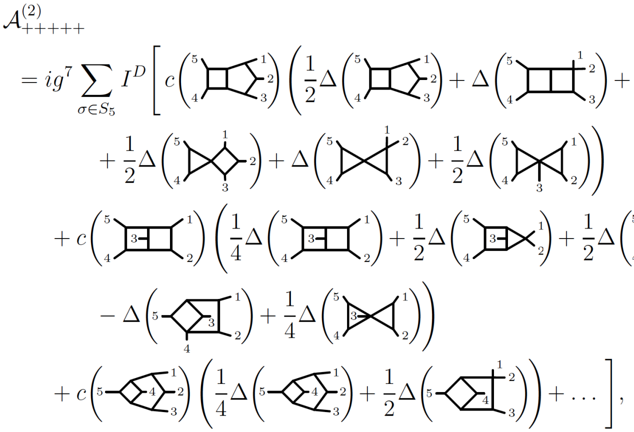
$$\mathcal{A}_n^{(0)} = g^{n-2} \sum_{\sigma \in S_{n-2}} c\left(\text{1 —} \sigma(2)\,\sigma(3) \cdots \sigma(n-1) \text{— n} \right) A^{(0)}(1\,\sigma(2)\cdots\sigma(n-1)n)$$

- Build color-dressed numerators out of corresponding cuts:



Multi-peripheral color decomposition (2)

[Badger, G.M., Ochirov & O'Connell '15]



(4.9)