

Some methods for calculations of Feynman diagrams

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Main reference: D.I. Kazakov,

"Analytical methods for Multiloop
calculations: Two Lectures on
the Method of Uniqueness",

JINR preprint JINR-E2-84-479

Dubna (1984)

① Definition: Euclidean space, $D = 4 - 2\varepsilon$, $\lambda = \frac{D}{2} - 1 = 1 - \varepsilon$

dimensional regularization
 $d^4k \rightarrow d^Dk (\mu^2)^\varepsilon$

$$\vec{p} \quad \xrightarrow{\alpha} \quad \equiv \quad \frac{1}{[p^2]^\alpha} \equiv \frac{1}{p^{2\alpha}}$$

$$\vec{p} \quad \xrightarrow[\alpha]{m^2} \quad \equiv \quad \frac{1}{[p^2 + m^2]^\alpha}$$

II Tadpoles

1.  $\equiv$

$$\int \frac{d^D k}{k^{2\beta} [k^2 + m^2]^\alpha} = (*)$$

polar. coordinates

$$d^D k = \frac{1}{2} (k^2)^{\frac{D}{2}-1} dk^2 dR$$

$$\int dR = S_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}$$

is the surface of the unit hypersphere in R^D

↑ Euler Γ -function

$$(*) \equiv [k^2 = m^2 z] = \frac{S_D}{2} \int_0^\infty \frac{dz \cdot z^{\lambda-\beta}}{(z+1)^\alpha} \frac{1}{(m^2)^{\alpha+\beta-\frac{D}{2}}} = (x)$$

Euler $\hat{B}$ -function : $\hat{B}(\beta, \alpha) = \frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\beta+\alpha)}$

$$\int_0^\infty \frac{dz \cdot z^\gamma}{(1+z)^\alpha} = \int_0^\infty \frac{dt \cdot (t-1)^\gamma}{t^\alpha} = \int_0^1 dp \, p^{\alpha-\gamma-2} (1-p)^\gamma = \hat{B}(\alpha-\gamma-1, \gamma+1)$$

$z+1=t, t=\frac{1}{p}$

Standard form

$$(x) = \frac{\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \frac{\Gamma(\frac{D}{2}-\beta) \Gamma(\alpha+\beta-\frac{D}{2})}{\Gamma(\alpha)} \frac{1}{(m^2)^{\alpha+\beta-\frac{D}{2}}}$$

$\hat{B}(\beta, \alpha)$

(A)

2. If $d+p < d_L$



Dimension regularization (convention):

(A1)

$$p \text{ tadpole} \stackrel{d}{m^2} \equiv 0 \quad \text{for any } d \text{ and } p$$

$m^2 \neq 0$

$[d+p=d_L, \delta\text{-function}; d \rightarrow d+\delta]$ in real calculations

$$\propto \delta(\beta+d-d_L)$$

S.G. Gorishnii, A.P. Isaev
Theor. Math. Phys. 62 (1985) 232

3. Consider the integral

$$\int \frac{d^D k}{[k^2 - 2p \cdot k + m^2]^\alpha} = (*)$$

Introduce $k_1 = k - p \Rightarrow [] = k_1^2 + \underbrace{m^2 - p^2}_{m_1^2}$

Moreover, $d^D k = d^D k_1$

$$(*) = \int \frac{d^D k_1}{[k_1^2 + \underbrace{m^2 - p^2}_{m_1^2}]^\alpha} = \pi^{\frac{D}{2}} B(0, \alpha) \frac{1}{[\underbrace{m^2 - p^2}_{m_1^2}]^{\alpha - \frac{D}{2}}} \quad (A2)$$

↑ "effective mass" (very important for future calculations)

III Feynman parametrization (here $[k^2 + m^2] \equiv [K]$)

(5)

$$\frac{1}{\prod_{i=1}^N [K_i]^{d_i}} = \frac{\Gamma(\sum_{i=1}^N d_i)}{\prod_{i=1}^N \Gamma(d_i)} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \dots \int_0^{1-\sum_{i=1}^{N-2} x_i} dx_{N-1} \cdot \frac{\prod_{i=1}^N x_i^{d_i-1}}{\left(\sum_{i=1}^N x_i [K_i] \right)^{\sum_{i=1}^N d_i}} \quad (B)$$

$x_N = 1 - \sum_{i=1}^{N-1} x_i$

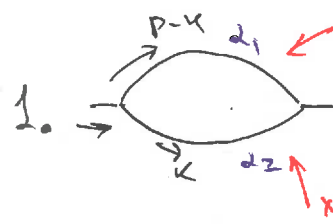
When $N=2$

$$\frac{1}{[K_1]^{d_1} [K_2]^{d_2}} = \frac{\Gamma(d_1 + d_2)}{\Gamma(d_1) \Gamma(d_2)} \int_0^1 dx_1 \frac{x_1^{d_1-1} (1-x_1)^{d_2-1}}{\left(x_1 [K_1] + (1-x_1) [K_2] \right)^{d_1 + d_2}} \quad (B1)$$

Feynman parametrization transform products of propagators to the new propagator with the effective mass, depending on x_i .
For this new propagator, we can apply our formulas for tadpoles.

IV One loop (massless loop)

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1.  $\xrightarrow{\text{Feyn. par.}} \frac{\Gamma(d_1 + d_2)}{\Gamma(d_1)\Gamma(d_2)} \int_0^1 dx \cdot x^{d_1-1} (1-x)^{d_2-1} \int \frac{d^D k}{[]^{d_1+d_2}} = \otimes$

$x_1 \equiv x$
 $x_2 \equiv 1-x$

$[] = x_0 (k-p)^2 + (1-x)k^2 = k^2 + p^2 x - 2(pk)x_0$ $\leftarrow$ Tadpole with $p \rightarrow p_1 = px$
 $m^2 = p^2 x$

$\int \frac{d^D k}{[]^{d_1+d_2}} \xrightarrow{\Sigma_4(A2)} \pi^{D/2} \frac{\Gamma(d_1+d_2-D/2)}{\Gamma(d_1+d_2)} \frac{1}{[\underbrace{p^2 x - \underbrace{p^2 x^2}_{m^2}}_{m_1^2}]_2}$ $[]_2 = p^2 x (1-x)$

$\otimes = \pi^{D/2} \frac{\Gamma(d_1+d_2-D/2)}{\Gamma(d_1)\Gamma(d_2)} \frac{1}{(p^2)^{d_1+d_2-D/2}} \int_0^1 \frac{dx \cdot x^{d_1-1} (1-x)^{d_2-1}}{[x(1-x)]^{d_1+d_2-D/2}} =$

$\int_0^1 dx \cdot x^{\tilde{d}_2-1} (1-x)^{\tilde{d}_1-1} = \frac{\Gamma(\tilde{d}_2)\Gamma(\tilde{d}_1)}{\Gamma(\tilde{d}_1+\tilde{d}_2)}$

$\tilde{d}_i = d_i - D/2$

$= \pi^{D/2} A(d_1, d_2) \frac{1}{(p^2)^{d_1+d_2-D/2}}$ $(\leftarrow)$

$\uparrow$
 G

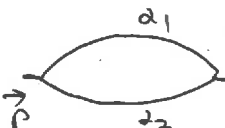
$A(d_1, d_2) = \frac{a(d_1)a(d_2)}{a(d_1+d_2-D/2)}, \quad a(d) = \frac{\Gamma(\tilde{d})}{\Gamma(d)}$

2. Recover Eq. (c) using Fourier transform

$$\int \frac{d^D p e^{ipx}}{p^{2d}} = \frac{\pi^{\frac{D}{2}} 2^{\frac{2\tilde{d}}{D-2d}} a(d)}{x^{2\tilde{d}}}$$

$$\int \frac{d^D x e^{-ipx}}{x^{2d}} = \frac{\pi^{\frac{D}{2}} 2^{2\tilde{d}} a(d)}{p^{2\tilde{d}}} \Rightarrow \frac{1}{p^{2d}} = \frac{1}{\pi^{\frac{D}{2}} 2^{2\tilde{d}} a(d)} \int \frac{d^D x e^{-ipx}}{x^{2\tilde{d}}}$$

Then



$$= \int \frac{d^D k}{k^{2d_1} (p-k)^{2d_2}} = \int d^D k \frac{1}{\pi} \frac{a(d_1) a(d_2)}{2^{2(d_1+d_2)}} \int \frac{d^D x_1 e^{-ikx_1}}{x_1^{2\tilde{d}_1}} \int \frac{d^D x_2 e^{-i(p-k)x_2}}{x_2^{2\tilde{d}_2}} =$$

$$\int d^D k e^{ik(x_2-x_1)} = (2\pi)^D \delta^D(x_2-x_1)$$

$$= \frac{2^D a(d_1) a(d_2)}{2^{2(d_1+d_2)}} \int \frac{d^D x_1 e^{-ipx_1}}{x_1^{2(\tilde{d}_1+\tilde{d}_2)}} = \frac{\pi^{\frac{D}{2}} 2 a(d_1) a(d_2)}{a(d_1+d_2-2\frac{D}{2})} \frac{1}{(p^2)^{d_1+d_2-\frac{D}{2}}}$$

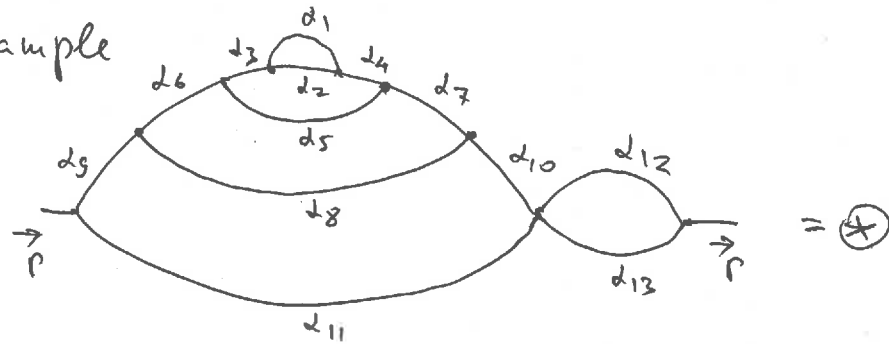
$$\frac{\pi^{\frac{D}{2}} 2^{2+\cancel{2(d_1+d_2)}} a(d_1+d_2)}{(p^2)^{-(d_1+d_2)+\frac{D}{2}}}$$

$$\tilde{d}_1 + \tilde{d}_2 = d_1 + d_2 - \frac{D}{2}$$

$$d_2 - (\tilde{d}_1 + \tilde{d}_2) = d_1 + d_2 - \frac{D}{2}$$

OK

4. Example



But ~~at two-loop~~ starting at two-loop there are diagrams like , which is not a combination of loops and chains

Steps

$$\textcircled{1} \quad \text{Diagram with } d_{12} \text{ and } d_{13} \text{ lines} = \pi^{\frac{D}{2}} A(d_{12}, d_{13}) \xrightarrow{d_{12}+d_{13}-D/2}$$

$$\textcircled{2} \quad \text{Diagram with } d_1, d_2, d_3, d_4, d_5 \text{ lines} = \pi^{\frac{D}{2}} A(d_1, d_2) \text{Diagram with } d_3, d_4, d_5 \text{ lines} = \pi^D A(d_1, d_2) A(\sum_{i=1}^4 d_i - \frac{D}{2}, d_5) \xrightarrow{\sum_{i=1}^5 d_i - D}$$



$$\textcircled{3} \quad \text{Diagram with } d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8 \text{ lines} = \pi^D A(d_1, d_2) A(\sum_{i=1}^4 d_i - \frac{D}{2}, d_5) \text{Diagram with } d_6, d_7, d_8 \text{ lines} = \pi^{\frac{3D}{2}} A(d_1, d_2) A(\sum_{i=1}^4 d_i - \frac{D}{2}, d_5) A(\sum_{i=1}^7 d_i - D, d_8) \xrightarrow{\sum_{i=1}^8 d_i - \frac{3D}{2}}$$

$$\textcircled{4} = \pi^{\frac{5D}{2}} A(d_1, d_2) A(\sum_{i=1}^4 d_i - \frac{D}{2}, d_5) A(\sum_{i=1}^7 d_i - D, d_8) A(\sum_{i=1}^{10} d_i - \frac{3D}{2}, d_{11}) \cdot A(d_{12}, d_{13}) \xrightarrow{\sum_{i=1}^{13} d_i - \frac{5D}{2}}$$

number of loops

5. To find the real result ('numbers') it is necessary to use the relation $\Gamma(d+1) = d\Gamma(d)$ to transform of Γ -function to the form $\Gamma(1+\epsilon)$ and later do use

$$\Gamma(1+\epsilon) = \exp \left[-\gamma_E \epsilon + \sum_{k=2}^{\infty} \frac{\gamma(k)}{k} (-1)^k \epsilon^k \right]$$

Note: the term $\pi^{D/2}$ ← number of integration can be not collected exactly, because any integration

gives:

$$\frac{\pi^{D/2} \Gamma(1+\epsilon)}{(2\pi)^D} \cdot g^2 (\mu^2)^\epsilon = \frac{g^2 \Gamma(1+\epsilon)}{(2\pi)^{D/2}} (\mu^2)^\epsilon = \frac{g^2}{16\pi^2} \underbrace{(4\pi)^\epsilon \Gamma(\epsilon)}_{(\bar{\mu}^2)^\epsilon} (\mu^2)^\epsilon$$

from measure of FI $\rightarrow (2\pi)^D$

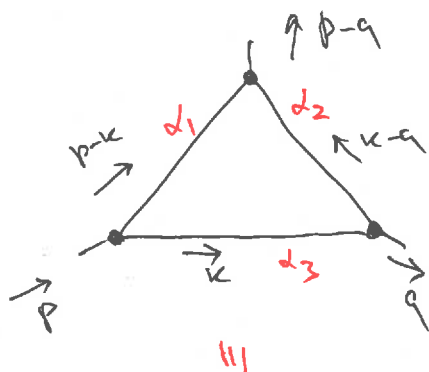
$\uparrow$ charge $\uparrow$ dim. reg. mass parameter

$\uparrow$ coupling constant

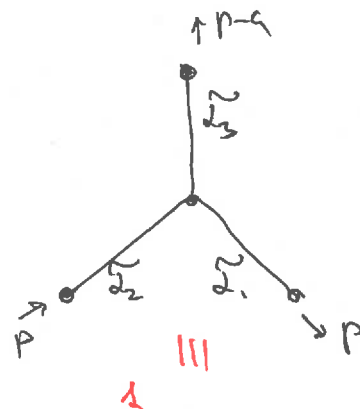
$(\bar{\mu}^2)^\epsilon$ new dim. scale $\equiv \bar{MS}$ -scheme

V Uniqueness relation: [M. D'Ermo, L. Peliti, G. Parisi, 1971]
 [A.N. Vassiliev, Yu. M. Pis'mak, Yu. R. Honkonen, 1981]

(11)



$$\sum d_i = 2 \Rightarrow \pi^2 A(d_1, d_2)$$



[?] Dissertation of A.M. Polyakov, 1977? ?]

$$\int \frac{d^D k}{k^{2d_3} [(p-k)^2]^{d_1} [(k-q)^2]^{d_2}} = \text{circled X}$$

$$\frac{1}{[(p-q)^2]^{d_3} p^{2d_2} q^{2d_1}}$$

To prove: inversion $(k \rightarrow \frac{1}{k}, p \rightarrow \frac{1}{p}, \frac{1}{q}) \subset$ conformal transformation (angles are kept $\equiv$ scalar products)

$$k^2 \rightarrow \frac{1}{k^2}, (p-k)^2 = p^2 - 2(p \cdot k) + k^2 \rightarrow \frac{1}{p^2} - \frac{2(p \cdot k)}{p^2 k^2} + \frac{1}{k^2} = \frac{(p-k)^2}{p^2 k^2}, d^D k = \frac{1}{2} (k^2)^{\frac{D}{2}-1} dk^2 d\Omega \rightarrow \frac{d^D k}{(k^2)^D}$$

$$\text{circled X} = \int \frac{d^D k_1 k_1^{2d_3} (k_1^2 p_1^2)^{d_1} (k_1^2 q_1^2)^{d_2}}{(k_1^2)^{d_3} [(p_1 - k_1)^2]^{d_1} [(k_1 - q_1)^2]^{d_2}}$$

loop in the new variables

Examples

(12)

$$\begin{aligned}
 & \text{Diagram 1: A circle with a vertical line through the center. The top arc is labeled } 1-\epsilon, \text{ the bottom arc is labeled } 1-\epsilon, \text{ and the vertical line is labeled } 2. \text{ It is enclosed in a dashed green circle.} \\
 & = \pi^{\frac{3}{2}\epsilon} A(1-\epsilon, 1-\epsilon) \cdot \text{Diagram 2} = \pi A(1\epsilon, 1\epsilon) A(2, 2) \cdot \text{Diagram 3}
 \end{aligned}$$

Diagram 2: A diamond shape with a horizontal line through the center labeled $-\epsilon$. Below it is a red loop labeled 2 .

Diagram 3: A horizontal line segment labeled 2 .

$$\begin{aligned}
 & \text{Diagram 4: A circle with a vertical line through the center. The top arc is labeled } 1-\epsilon, \text{ the bottom arc is labeled } 1-\epsilon, \text{ and the vertical line is labeled } -\epsilon. \text{ It is enclosed in a dashed green circle.} \\
 & = \frac{1}{h^{\frac{3}{2}\epsilon}} \perp A(1\epsilon, 1\epsilon) \cdot \text{Diagram 5} = \frac{\pi^{\frac{3}{2}} A^2(1\epsilon, 1\epsilon)}{A(1\epsilon, 1-\epsilon)} \cdot \text{Diagram 6} = \\
 & = \pi^{\frac{3}{2}} \frac{A^2(1\epsilon, 1\epsilon)}{A(1\epsilon, 1\epsilon)} A(2, 2\epsilon) \cdot \text{Diagram 7}
 \end{aligned}$$

Diagram 5: A circle with a vertical line through the center. The top arc is labeled 1ϵ , the bottom arc is labeled 2 , and the vertical line is labeled 1ϵ .

Diagram 6: A circle with a vertical line through the center. The top arc is labeled 2ϵ , the bottom arc is labeled 2 , and the vertical line is labeled ϵ . It is enclosed in a red bracket.

Diagram 7: A horizontal line segment labeled ϵ .

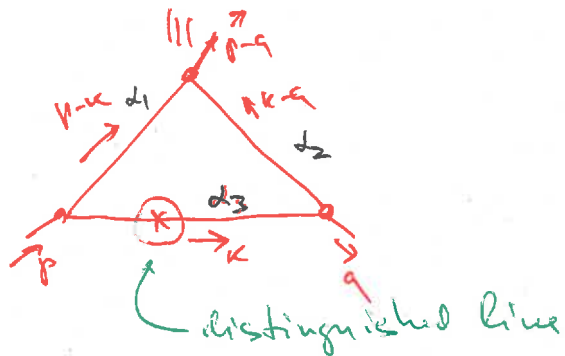
VI IBP (integration by parts) procedure

[K.G. Chetyrkin, F.V. Tkachov, 1981] $\rightarrow p$ -space
 [A.N. Vassiliou, et al. 1981] $\rightarrow x$ -space

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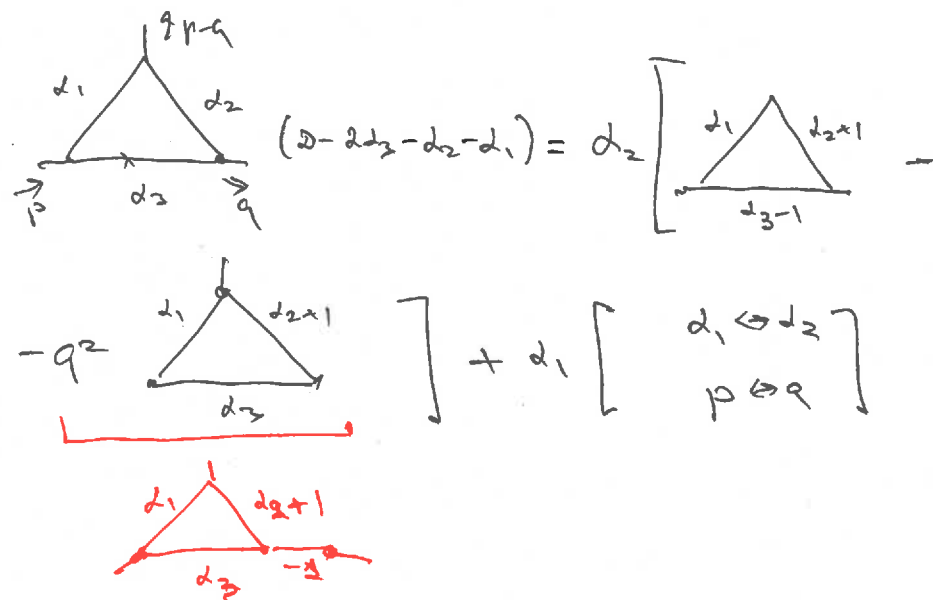
$$\int \frac{d^D k}{[k^{2d_3} (k-q)^{2d_2} (k-p)^{2d_1}]} = \int d^D k \left[\underbrace{\frac{1}{d_3 k} \left(\frac{k^H}{[]_+} \right)}_{=0} - k^H \frac{1}{d_3 k} \frac{1}{[]_+} \right]$$

=0 Stokes theorem



$$-k^H \frac{1}{d_3 k} \frac{1}{k^{2d_3}} = \frac{2d_3}{k^{2d_3}}$$

$$-k^H \frac{1}{d_3 k} \frac{1}{(k-q)^{2d_2}} = 2d_2 \frac{(k, k-q)}{(k-q)^{2(d_2+1)}} = d_2 \frac{k^2 + (k-q)^2 - q^2}{(k-q)^{2(d_2+1)}}$$



Example:

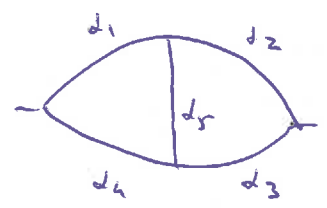
14

$$\rightarrow \text{Diagram} (D-2-d_1-d_2) = d_1 \left[\text{Diagram 1} - \text{Diagram 2} \right] + d_2 \left[d_1 \leftrightarrow d_2 \right] =$$



$$= \left\{ d_1 \frac{A(1,1)}{A(1,1)} \left(A(d_1+1, d_2) - A(d_1-1, d_2+1) \right) + \left[d_1 \leftrightarrow d_2 \right] \right\} \frac{1}{(p^2)^{d_1+d_2-1+2\epsilon}}$$

VII Transformations. Relations between FI

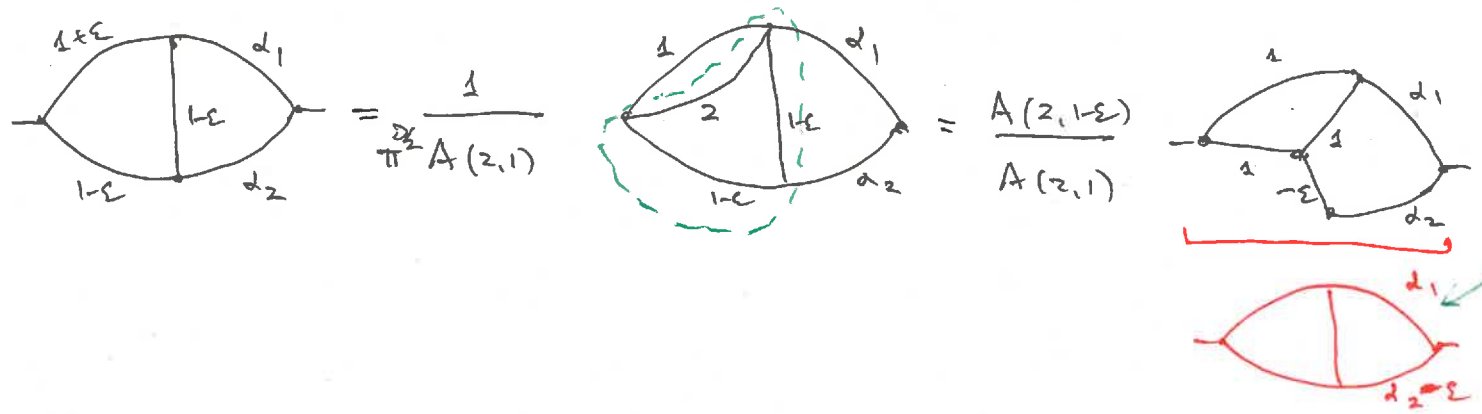


with different d_i .

[A.N. Vassiliev et. al., 1981]
 [S.G. Goussarov, A.P. Isaev, 1985]
 [D.T. Barfoot, D.J. Broadhurst, 1988]

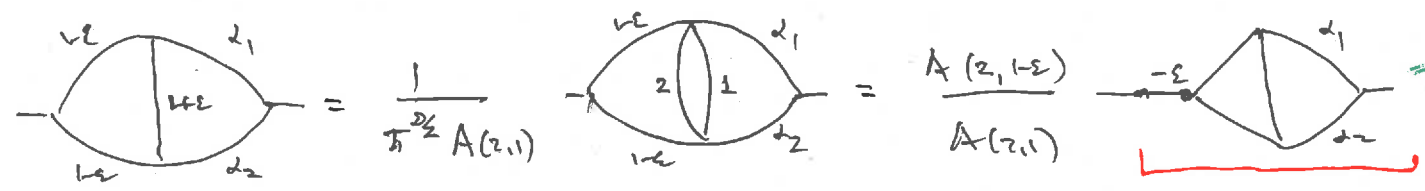
Examples

(I) The ~~lateral~~ additional loop $\rightarrow$ to "lateral" line $\rightarrow$ very big group of transformations



calculated before by IRP

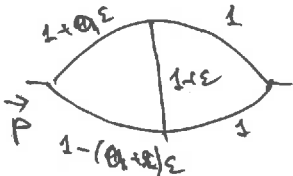
(II) The additional loop $\rightarrow$ to vertical line



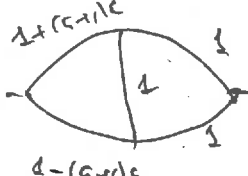
calculated before by IRP

(III) Additional propagator

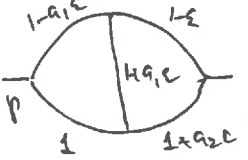
16

Consider  = $\frac{\epsilon}{(p^2) 1\epsilon} \Rightarrow \frac{\epsilon}{p^2} = \text{diagram with a cross on the left line} \Rightarrow \text{diagram with a cross on the left line and a dashed green loop} = \text{diagram with a cross on the left line and a dashed green loop and a dashed green vertical line} \xrightarrow{\frac{1}{\epsilon^{\frac{D}{2}} A(2-2\epsilon, 1+(q_1+1)\epsilon)}} =$

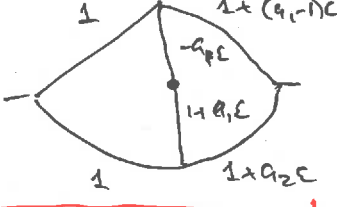
$= \frac{A(2-2\epsilon, 1+\epsilon)}{A(2-2\epsilon, 1+(q_1+1)\epsilon)} \text{diagram with a cross on the left line and a dashed green loop} \leftarrow \text{Calculated by IBP}$

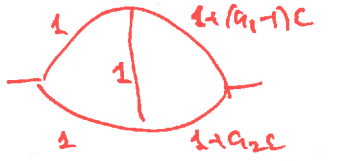


(IV) Additional loop

Consider  = $\frac{\epsilon}{(p^2)^{1+(q_1+q_2+1)\epsilon}}$ Integrate with additional propagator:

$\epsilon A(1+(q_2+1)\epsilon, 2+(q_1-1)\epsilon) \xrightarrow{\frac{1}{p^2(1+(q_1+q_2-1)\epsilon)}} = \text{diagram with a cross on the left line and a dashed green loop and a dashed green vertical line} = A(1-q_1\epsilon, 1\epsilon) \text{diagram with a cross on the left line and a dashed green loop and a dashed green vertical line} = \text{diagram with a cross on the left line and a dashed green loop and a dashed green vertical line and a dashed green horizontal line} \xrightarrow{\text{Calculated by IBP}} \text{diagram with a cross on the left line and a dashed green loop and a dashed green vertical line and a dashed green horizontal line}$





⑤

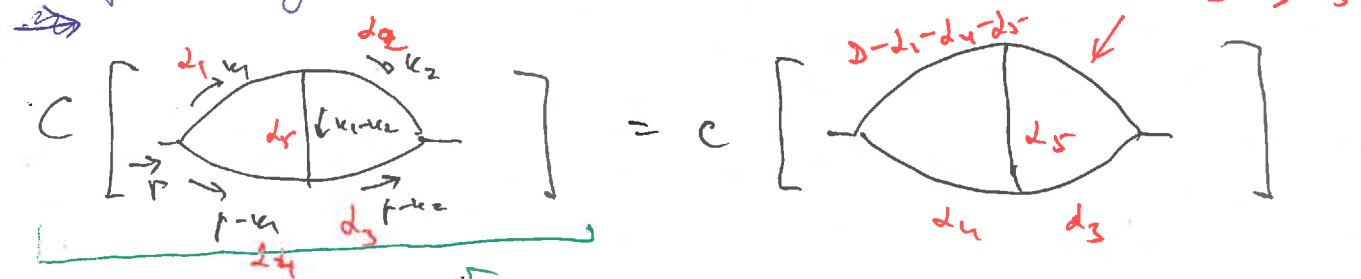
Conformal transformation of inversion

16a

All $k_i^h \rightarrow k_i^h \Rightarrow k_i^2 \Rightarrow \frac{1}{k_i^2}$ $(k_i - k_j)^2 \Rightarrow \frac{(k_i - k_j)^2}{k_i^2 k_j^2}$ (see also prove of uniqueness relation), $d^2 k_i \rightarrow \frac{d^2 k_i}{(k_i^2)^2}$

$$\left(\frac{d^2 k_1 d^2 k_2}{k_1^{2D} k_2^{2D} (k_1 - k_2)^{2D} (p - k_1)^{2d_4} (p - k_2)^{2d_3}} \right) \rightarrow \left(\frac{d^2 k_1 d^2 k_2 k_1^{2D} k_2^{2D} (k_1^2 k_2^2)^{2D} (p^2 k_1^2)^{2d_4} (p^2 k_2^2)^{2d_3}}{k_1^{2D} k_2^{2D} (k_1 - k_2)^{2D} (p - k_1)^{2d_4} (p - k_2)^{2d_3}} \right)$$

Graphically



coefficient function: $I = C[I] \frac{1}{p^{2D}}$

Example

$$C \left[\begin{array}{c} d_1 \quad 1-2\epsilon \\ \text{diagram} \\ d_4 \quad 1 \end{array} \right] = C \left[\begin{array}{c} \bar{d}_1 \quad 1 \\ \text{diagram} \\ d_4 \quad 1 \end{array} \right], \text{ where } \bar{d}_1 = D-1-d_1-d_4$$

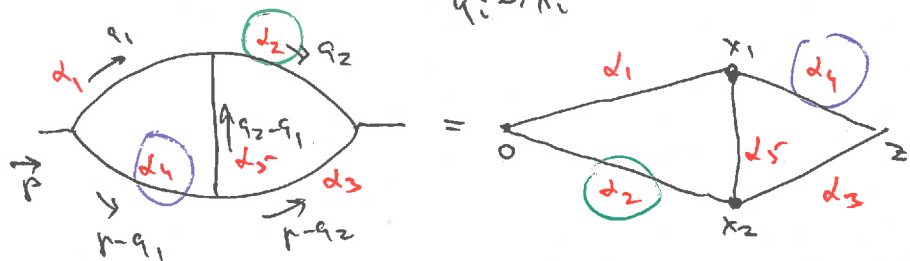
↑ calculated by IBP

VIII Dual diagrams and Fourier transform

(17)

In principle we can work in x^4 space. To transform FI to x^4 space we can use two possibilities

(A) Dual diagram: $p \rightarrow z$
 $q_i \rightarrow x_i$



Integral is same but graphical representation is different.

For this case only the lines with d_2 and d_4 are exchanged.

Calculations in QCD [D.I. Kazakov, A.V.K., 1988, 1990]
Method of Gegenbauer Polynomials [A.V.K., 1996]
N=4 SYM [J.M. Drummond, J. Henn, G.P. Korchemsky,
E. Sokatchev, 2010]

(B) Fourier transform

$$\text{Bubble diagram} = \frac{C}{(p^2)^{d_F}} \quad d_F = \sum_{i=1}^5 d_i - D, \quad \tilde{d}_i = D - d_i$$

If we do Fourier transform of lhs and rhs parts we have

$$C = \frac{\prod_{i=1}^5 a(x_i)}{a(\tilde{d}_F)} \quad \tilde{C} = \frac{1}{(x^2)^{\tilde{d}_F}}$$

Because $a(\tilde{d}) = 1/a(d)$

$$\tilde{C} = \frac{\prod_{i=1}^5 a(\tilde{d}_i)}{a(\tilde{d}_F)} \quad \text{Bubble diagram}$$

If we would like to continue to work in $\mathcal{P}$ space we can transform the initial diagram to $\mathcal{X}$ space by dual [Fourier] transform and to return by Fourier [dual] transform:

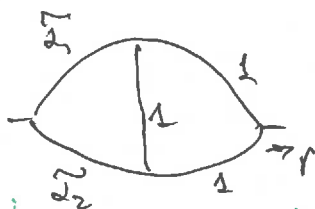
$$\text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} = \frac{\prod_1^5 a(d_i)}{a(d_P)} \text{Diagram 4}$$

In general (The diagram has N lines and L loops):

$$\frac{\prod_1^N a(d_i)}{a(d_P)}, \quad d_P = \sum_1^N d_i - \frac{L}{2} D$$

Example

$$\text{Diagram 1} = \frac{\prod_1^2 a(d_i) \cdot a^3(l_5)}{a(d_1 + d_2 - 1 \cdot 2)} \text{Diagram 2}$$

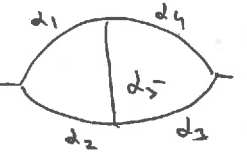


$$d_i = \frac{D}{2} - d_i$$

calculated by IBP

IX General result

(19)



$$d_i = 1 + q_i \varepsilon \quad \frac{\chi_2}{1-2\varepsilon} \left\{ A_0 q_3 + A_1 q_4 \varepsilon + A_2 q_5 \varepsilon^2 + [A_3 q_6 - A_4 q_3^2] \varepsilon^3 + O(\varepsilon^4) \right\}$$

[I. Bienenbaum, S. Weinzierl, 2003]
[Kazakov, 1985]

$$K_n = \exp \left[-n \left(\gamma \varepsilon + \frac{q(2)}{2} \varepsilon^2 \right) \right], \quad A_0 = 6, \quad A_1 = 9, \quad A_3 = \frac{5}{2} (A_2 - 6), \quad A_n = a_1^n + a_2^n + a_3^n + a_4^n$$

$$A_2 = 42 + 30A + 45a_5 + 10A_2^2 + 15a_5^2 + 15a_5A + 10(a_1a_2 + a_3a_4 + a_1a_4 + a_2a_5) + 5(a_1a_3 + a_2a_4)$$

$$A_4 = 46 + 42A + 45a_5 + 14A_2 + 15a_5^2 + 33a_5A + 50(a_1a_2 + a_3a_4) + 31(a_1a_3 + a_2a_4) + 14(a_1a_4 + a_2a_5)$$

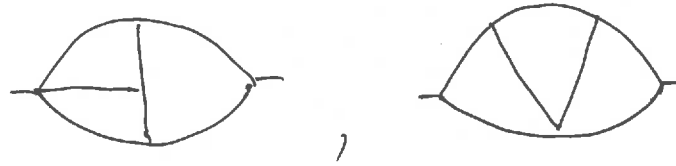
$$+ 6a_5A_2 + 6a_5^2A + 24a_5(a_1a_2 + a_3a_4) + 12a_5(a_1a_3 + a_2a_4) + 12(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4)$$

$$+ 12(a_1^2a_2 + a_2^2a_1 + a_3^2a_1 + a_4^2a_3) + 6(a_1^2a_3 + a_3^2a_1 + a_3^2a_4 + a_4^2a_2)$$

X The-loop case

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Two basic topology



(A) IBP $(D-4) = 2 \cancel{\text{triangle}} - \cancel{\text{triangle}} - \text{bubble}^2 = A(z,1) \text{triangle}^{1+\epsilon} -$

$\text{bubble} = \frac{e}{p^2(\mu^2\epsilon)}$
 $\text{triangle} = e \text{triangle}^{\mu^2\epsilon} = e A(z, \mu^2\epsilon) \frac{1}{(p^2)^{1+\epsilon}}$

$- A(z, \mu^2\epsilon) \text{triangle}^{\epsilon}$

So we transform 3-loop diagrams to two loop ones.

Kazakov :

$\stackrel{d_i = 1 + a_i \epsilon}{=} \frac{K_3}{1-2\epsilon} \left[20 \epsilon_5 + \epsilon \left\{ 50 \epsilon_6 + (20 + 6 \sum_4^7 a_i) \epsilon_3^2 \right\} + O(\epsilon^2) \right]$

[D.I. Kazakov, 1985]

Consider

$$\begin{aligned}
 & \text{Diagram 1: A sphere with four vertices labeled } d_1, d_2, d_3, d_4 \text{ in red. Internal lines are labeled } d_5, d_6, d_7. \text{ An arrow } \vec{p} \text{ points into the left vertex.} \\
 & \equiv \frac{\prod_{i=1}^7 a(d_i)}{a(\sum_{i=1}^7 d_i - \frac{7}{2}D)} \left[\begin{aligned} & \text{Diagram 2: Similar to Diagram 1, but with vertices labeled } \tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_4 \text{ and internal lines } \tilde{d}_5, \tilde{d}_6, \tilde{d}_7. \text{ An arrow } \vec{p} \text{ points into the left vertex.} \\ & \equiv \text{Diagram 3: A sphere with four vertices labeled } \tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_4 \text{ and internal lines } \tilde{d}_5, \tilde{d}_6, \tilde{d}_7. \text{ An arrow } \vec{p} \text{ points into the right vertex.} \end{aligned} \right] = \text{Diagram 3}
 \end{aligned}$$

it is considered already

$$\begin{aligned}
 & d_i = 1 + a_i \varepsilon \\
 & \tilde{d}_i = 1 - (a_i + 1) \varepsilon \\
 & \frac{k_3}{(1-2\varepsilon)} \left[20 \zeta_5 + \varepsilon \left\{ 50 \zeta_6 - (4 + 6 \sum_{i=1}^7 a_i) \zeta^2(\varepsilon) \right\} + O(\varepsilon^2) \right]
 \end{aligned}$$

obtained from the previous one: $a_i \rightarrow -(1+a_i)$

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Lecture 3

22

23

To calculate anomalous dimensions and β -functions really, it is necessary to evaluate (often) only the singular structure of diagrams. It is often gives a possibility to obtain results at high orders. The important property: the

absence of p -dependence in KR' of the diagrams (here = in singularities of the diagram)

Examples

$$\textcircled{1} \quad \text{Sing} \left[\begin{array}{c} \text{Diagram 1} \\ \hline \text{Diagram 2} \end{array} \right] = \text{Sing} \left[\text{Diagram 3} \right] = \textcircled{5}$$

$$\text{Diagram 4} = c \xrightarrow{u2c} \text{Diagram 5} = c \xrightarrow{u2c} \text{Diagram 6} = c A(1, u2c) \cdot 3\epsilon$$

$$A(1, u2c) = \frac{\Gamma(1-\epsilon) \Gamma(1-\epsilon) \Gamma(2-\epsilon)}{\Gamma(1) \Gamma(2-\epsilon) \Gamma(2-\epsilon)} = \frac{1}{3\epsilon} + \text{finite}$$

$$\textcircled{5} = \frac{1}{3\epsilon} \underbrace{\text{finite}}_{62(13)} \text{Diagram 7} = \frac{2}{\epsilon} 2(13)$$

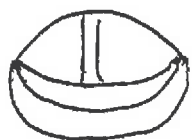
② $\text{Sing} \left[\begin{array}{c} \text{Diagram 1} \\ \parallel \\ \text{Diagram 2} \end{array} \right] = \text{Sing} \left[\begin{array}{c} \text{Diagram 3} \end{array} \right] = \text{finite} \underbrace{\begin{array}{c} \text{Diagram 3} \\ \hline 20 \zeta(5) \end{array}}_{\text{finite}} = \frac{5}{2} \zeta(5)$

~~22~~
23

$\text{Diagram 4} = c \xrightarrow{2+2\epsilon} \text{Diagram 5} = c \underbrace{\text{Diagram 6}}_{\substack{= c A(1,4,2) \frac{4c}{\epsilon} \\ \approx \frac{c}{4\epsilon} \times \text{finite}}} = c A(1,4,2) \frac{4c}{\epsilon}$

β -function (5-loop) in ϕ^4 -model in $D=4$ Chetyrkin et al., Phys. Lett. B 132 (1983) 351

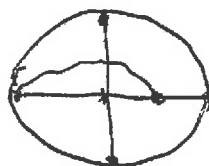
4-diagrams have been calculated numerically



(a)



(b)



(c)



(d)

① $\text{Sing} \left[\text{Diagram} \right] = \mathcal{D}$

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Let $\text{Diagram} = C \xrightarrow{8 - \frac{7}{2}\epsilon = 2+3\epsilon}$

$\text{Diagram} = C \xrightarrow{2+3\epsilon} \text{Diagram} = C \xrightarrow{2+3\epsilon} \text{Diagram} = C \xrightarrow{2+3\epsilon} \text{Diagram} \xrightarrow{5\epsilon}$

$\xrightarrow{2} = \frac{\Gamma^2(1-\epsilon) \Gamma(2-\epsilon) \Gamma(2+3\epsilon) \Gamma(-\epsilon) \Gamma(5\epsilon)}{\Gamma(1) \Gamma(2) \Gamma(2) \Gamma(2+3\epsilon) \Gamma(2-6\epsilon)} = -\frac{1}{20\epsilon^2} \frac{1}{(1+3\epsilon)(1-6\epsilon)} \frac{\Gamma^2(1-\epsilon) \Gamma(4\epsilon) \Gamma(1+5\epsilon)}{\Gamma(1+3\epsilon) \Gamma(1-6\epsilon)} \Gamma^2(1-\epsilon) (1+O(\epsilon^2))$

$\mathcal{D} = \text{Sing} \left[-\frac{1}{20\epsilon^2} \frac{1}{(1+3\epsilon)(1-6\epsilon)} \cdot \left\{ \text{Diagram} \text{ with the accuracy } O(\epsilon) \right\} \right]$

Consider Diagram

IBP

IRP

$$\text{Diagram} (2,1) = 2 \left[\text{Diagram}_1 - \text{Diagram}_2 \right] = 0$$

$A(2,1) \xrightarrow{uc}$

~~24~~
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Consider

$$\begin{aligned} \xrightarrow{p} \text{Diagram} (2,1) &= 2 \left[\text{Diagram}_1 - \text{Diagram}_2 - p^2 \text{Diagram}_3 \right] \\ \text{Diagram} (2,1) &= 2 \left[\text{Diagram}_1 - \text{Diagram}_2 \right] \end{aligned}$$

$$\Rightarrow \text{Diagram}_3 = \frac{1}{p^2} \left[4 \text{Diagram}_1 - 3 \text{Diagram}_2 \right] = \frac{A(1,2)}{p^2} \left[4 \text{Diagram}_{uc} - 3 \text{Diagram}_{uc} \right]$$

$$\text{Diagram} = \frac{2A(2,1)}{p^2} \left[\text{Diagram}_1 - 4 \text{Diagram}_2 + 3 \text{Diagram}_3 \right]$$

Table integrals

In principle,
where is full ϵ -dependence
(exact result) because

$$\text{Diagram}_1 \sim \epsilon_2^2 (\dots 1)$$

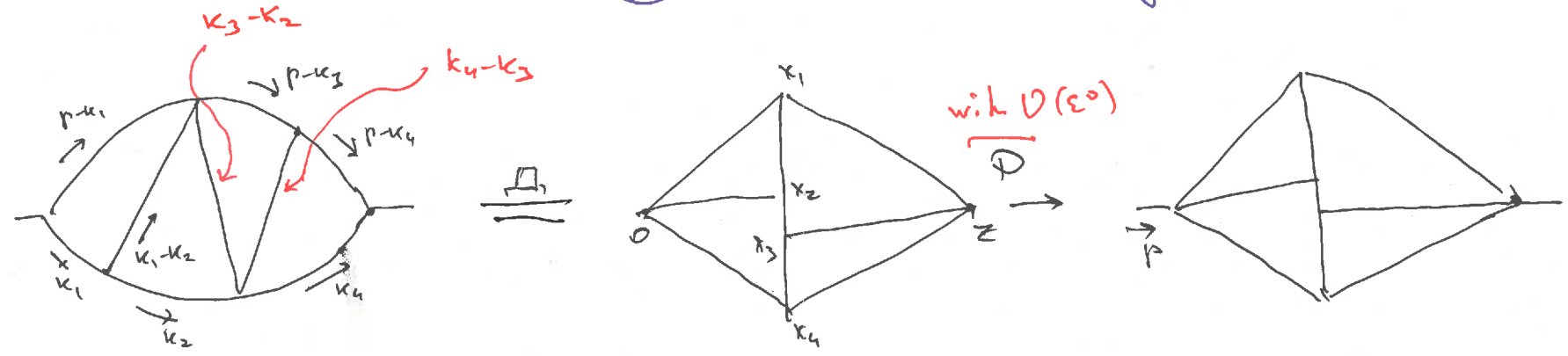
$$\text{Diagram}_1 \Big|_{p^2=1} = \frac{K_3}{1-2\epsilon} \left[20\epsilon_5 + [50\epsilon_6 + 44\epsilon_3^2]\epsilon + [317\epsilon_7 + 132\epsilon_2\epsilon_3]\epsilon^2 \right]$$

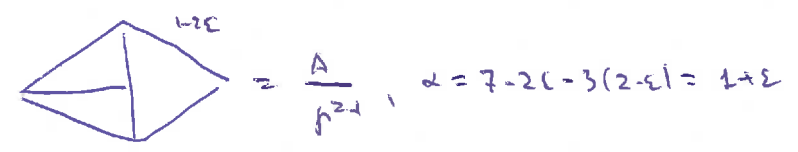
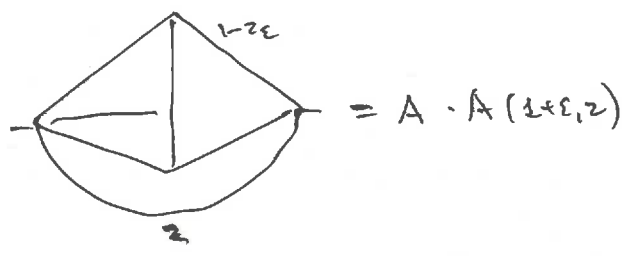
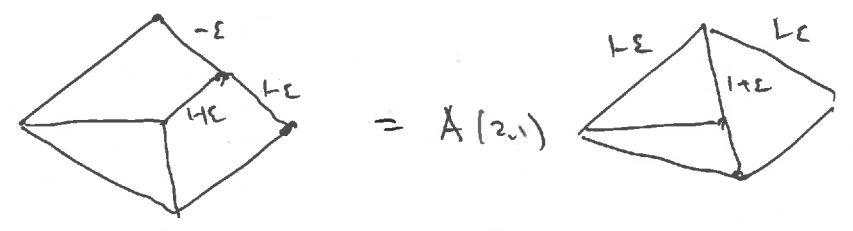
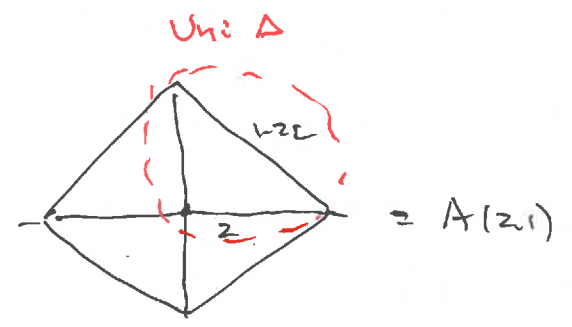
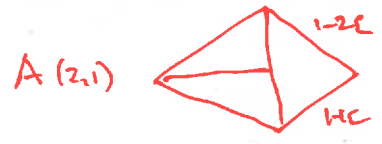
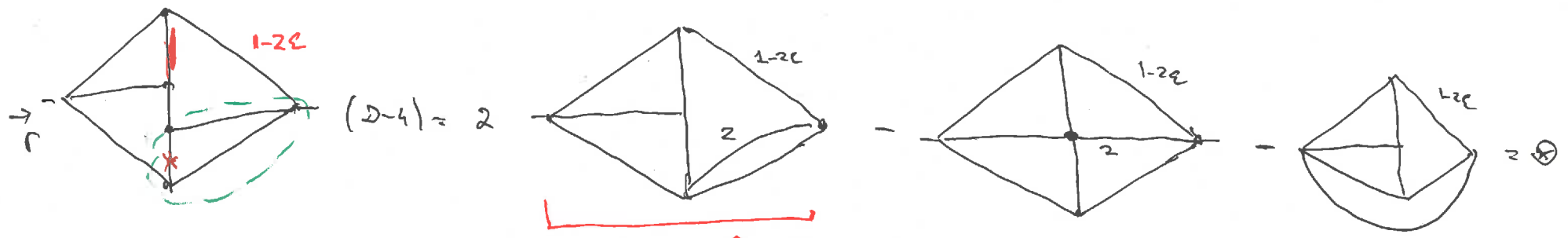
$K_h = \exp \left[-h \left(\gamma_2 \epsilon + \frac{\gamma_2}{2} \epsilon^2 \right) \right]$

Sing $\left[\text{Diagram of a sphere with four triangles} \right]$

Let $\text{Diagram of a sphere with four triangles} = \frac{A}{p^{2d}}$, $d = 9 - 4 \cdot \frac{9}{2} = 9 - 20 = -11$ $\Rightarrow$ $\text{Diagram of a sphere with four triangles} = A \cdot A(1, 1+4\epsilon) = \frac{A}{5\epsilon}$,

So we should find $\text{Diagram of a sphere with four triangles}$ with $O(\epsilon^0)$ accuracy





$$\begin{aligned} (*) &= A(2,1) \left[2 \cdot \text{Diamond}(1-2\varepsilon, 1+\varepsilon) - \text{Diamond}(1\varepsilon, 1+\varepsilon) - \right. \\ &\quad \left. - \frac{A(2,4\varepsilon)}{A(2,1)} \cdot \text{Diamond}(1-2\varepsilon) \right] = (*) \end{aligned}$$

$$A(z,1) = \frac{\Gamma(-\varepsilon) \Gamma(1-\varepsilon) \Gamma(1-\varepsilon)}{\Gamma(2) \Gamma(1) \Gamma(1-2\varepsilon)} = -\frac{1}{\varepsilon} \frac{\Gamma^2(\varepsilon) \Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

$$A(z,1+\varepsilon) = \frac{\Gamma(-\varepsilon) \Gamma(1-2\varepsilon) \Gamma(1-2\varepsilon)}{\Gamma(2) \Gamma(1-\varepsilon) \Gamma(1-3\varepsilon)} = -\frac{1}{\varepsilon} \frac{\Gamma(\varepsilon) \Gamma(1-\varepsilon) \Gamma(1-2\varepsilon)}{\Gamma(1-\varepsilon) \Gamma(1-3\varepsilon)}$$

$$\Rightarrow$$

$$\Rightarrow \frac{A(z,1+\varepsilon)}{A(z,1)} = \frac{\Gamma^2(1-2\varepsilon) \Gamma(1-2\varepsilon)}{\Gamma^2(1-\varepsilon) \Gamma(1-\varepsilon) \Gamma(1-3\varepsilon)} = \frac{\exp \left[\delta_{\varepsilon} \varepsilon \left(\cancel{2 \cdot 2 - 2} \right) + \sum_{h=2}^{\infty} \frac{\delta(h)}{h} \varepsilon^h \left[2 \cdot 2^h + (-2)^h \right] \right]}{\exp \left[\delta_{\varepsilon} \varepsilon \left(\cancel{2 \cdot (-1) + 3 \cdot 1} \right) + \sum_{h=2}^{\infty} \frac{\delta(h)}{h} \varepsilon^h \left[1 + 3^h + (-2)^h \right] \right]} =$$

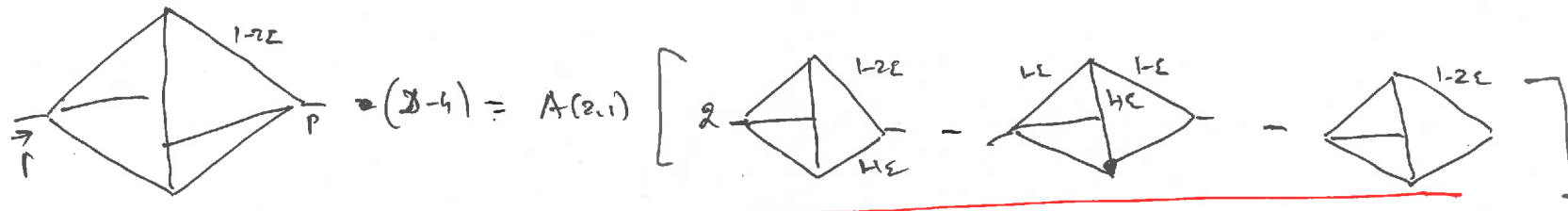
$$\Gamma(1-x) = \exp \left[-\gamma_x x + \sum_{h=2}^{\infty} \frac{\delta(h)}{h} (-x)^h \right]$$

$$= \exp \left[\sum_{h=2}^{\infty} \frac{\delta(h)}{h} \varepsilon^h p(h) \right] \approx \cancel{O(\varepsilon^2)} O(\varepsilon^3)$$

$$p(1) = 2^{1+1} - 1 - 3^1 + 2(-1)^1 [2^{1-1} - 1]$$

$$p(2) = 8 - 1 - 9 + 2 = 0$$

$$p(3) = \underline{16} - 1 - 27 - \underline{6} = 10 - 28 = -18$$



$$= (D-4) = A(2,1) \left[2 \text{ (triangle with } 1-2\epsilon, 4\epsilon) - \text{ (triangle with } 4\epsilon, 1-\epsilon) - \text{ (triangle with } 1-2\epsilon) \right]$$

Table integrals

After evaluation:  $\stackrel{O(\epsilon^2)}{=} \text{ (triangle with } 1-2\epsilon, 4\epsilon) = \frac{441}{8} \zeta_7 + O(\epsilon^2)$

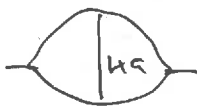
So $\text{Sing} \left[\text{ (triangle in circle) } \right] = \frac{441}{40} \frac{\zeta_7}{\epsilon} + \text{ (crossed out terms) } \quad \text{--- (see also, P.A. Baikov, K.G. Chetyrkin, ~~2000~~ 2010)}$

Really to ~~have obtain~~ obtain the result we should have longer expansion of the table integrals. ~~It is~~

The additional coefficients of expansion can be obtained from better knowledge of the diagram 

XII Functional Equations [Kazakov, 1985]

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Consider  $\equiv F(1+a)$

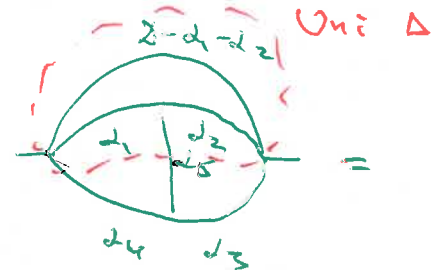
Remember: ①  $\xrightarrow{\text{inversion}} \langle \text{Diagram with } \bar{d}_1, \bar{d}_2, \bar{d}_3, \bar{d}_4 \text{ and } \bar{d}_5 \rangle$

$$\bar{d}_1 = D - d_1 - d_4 - d_5$$

$$\bar{d}_2 = D - d_2 - d_3 - d_5$$

② additional loop

$$\text{Diagram with } d_1, d_2, d_3, d_4 \text{ and } d_5 = \frac{1}{A(D - d_1 - d_2, \sum_{i=1}^5 d_i - D)}$$



$$= \frac{A(d_1, d_2)}{A_1} \left[\text{Diagram with } \bar{d}_1, \bar{d}_2, \bar{d}_3, \bar{d}_4 \text{ and } \bar{d}_5 \right] = \left[\text{Diagram with } \bar{d}_1, \bar{d}_2, \bar{d}_3, \bar{d}_4 \text{ and } \bar{d}_5 \right]$$

$$\text{Diagram with } 1+a \xrightarrow{\text{inver}} \text{Diagram with } \bar{d}_1, \bar{d}_2 \xrightarrow{\text{add. loop}} \text{Diagram with } \bar{d}_1, \bar{d}_2 \text{ and } \bar{d}_5 = 2(D - 1 - a - 2a) + 1 + a - 2 + \epsilon = 1 - a - 3\epsilon$$

$$\bar{d}_1 = D - 3 - a = 1 - a - 2\epsilon$$

$$D/2 - \bar{d}_1 = 1 + a + \epsilon$$

$$\bar{d}_2 = D - (1 + a + \epsilon) - \bar{d}_5 = D - 1 - (2 - 2\epsilon) = 1$$

$$\text{Diagram with } \bar{d}_1, \bar{d}_2 \xrightarrow{\text{inver}} \text{Diagram with } \bar{d}_1, \bar{d}_2 \text{ and } \bar{d}_5 = \text{Diagram with } 1 - a - 3\epsilon$$

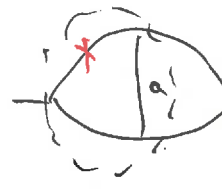
$$F(1+a) = F(1-a-3\epsilon) \quad (1)$$

IBP

(30)



$$\frac{(2-4-2a)}{-2(a+\epsilon)} = 2 \left[\text{bubble with vertical line and label } a \text{ (underlined green)} - \text{bubble with vertical line and label } 1+a \text{ (underlined green)} \right] \quad (2.1)$$



$$\frac{(2-3-a)}{1-2\epsilon-a} = \text{bubble with vertical line and label } a \text{ (underlined green)} - p^2 \text{bubble with vertical line and label } a \text{ (underlined green)} + a \left[\text{bubble with vertical line and label } a+1 \text{ (crossed out green)} - \text{bubble with vertical line and label } a+1 \text{ (crossed out green)} \right] \quad (2.2)$$

$$\sum = (2.1) + (2.2)$$

$$-(a+\epsilon) F(1+\epsilon) + (1-2\epsilon-a) F(a) = \text{bubble with vertical line and label } a \text{ (underlined green)} - \text{bubble with vertical line and label } a \text{ (underlined green)} = \underline{A(a, \epsilon) A(1, 1+\epsilon-\epsilon) - A(1, \epsilon+1) A(2, 1+\epsilon+\epsilon)}$$

F-functions

After little algebra

$$F(1+a) = \frac{1-2\epsilon-a}{a+\epsilon} F(a) + \frac{2(2a-1+3\epsilon) \Gamma(-a-\epsilon) \Gamma(a+2\epsilon) \Gamma^2(L\epsilon)}{(a+\epsilon) \Gamma(a+1) \Gamma(1-3\epsilon-a)}$$

(2)

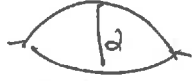
(31)

Solution of the system of equations (1) and (2):

$$F(1+a) = 2 \frac{\Gamma^2(a) \Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)} \left[\frac{\Gamma(a) \Gamma(a) \Gamma(-a-2\varepsilon) \Gamma(a+2\varepsilon)}{\Gamma(2\varepsilon) \Gamma(4\varepsilon)} + \frac{\Gamma(-a-\varepsilon) \Gamma(a+2\varepsilon)}{\Gamma(a) \Gamma(1-a-2\varepsilon)} \right] \times$$

$$\times \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+1-2\varepsilon)}{\Gamma(n+\varepsilon)} \left(\frac{1}{n+a+\varepsilon} + \frac{1}{n-a-2\varepsilon} \right)$$

Two hypergeometric functions ${}_3F_2(\dots, -1)$ with the argument -1 .

XIII) Another approach, useful for  is based on Gegenbauer Polynomials ^(B2)
 [K.G. Chetyrkin, A.L. Kataev, F.V. Tkachov (GP) 1980]
 [A.V.K. 1996]

GP gives a possibility to ~~cancel~~ one propagator of a diagram
 Here we will follow to

$$(1 - 2vt + v^2)^{-\epsilon} = \sum_{h=0}^{\infty} \underbrace{C_h^{\epsilon}(1)}_{\text{GP}} v^h \quad (v \leq 1) \leftarrow \text{definition}$$

Application for propagators

$$\frac{1}{[(p_1 - p_2)^2]^{\epsilon}} = \frac{1}{[p_1^2 - 2|p_1||p_2| \underbrace{C_0(\hat{p}_1 \hat{p}_2)}_{\hat{p}_1 \hat{p}_2} + p_2^2]^{\epsilon}} = \left[\frac{\theta(p_1^2 > p_2^2)}{(p_1^2)^{\epsilon}} \sum_{h=0}^{\infty} C_h^{\epsilon} \left(\underbrace{C_h(\hat{p}_1 \hat{p}_2)}_{\hat{p}_1 \hat{p}_2} \right) \left(\frac{p_2^2}{p_1^2} \right)^{\frac{h}{2}} \right] + [1 \leftrightarrow 2]$$

③ $\lambda = D/2 - 1 \leftarrow$ special choice

$$\int d\hat{p}_2 C_n^{\lambda}(\hat{p}_1 \hat{p}_2) C_n^{\lambda}(\hat{p}_2 \hat{p}_3) = \frac{1}{n+2\lambda} \delta_{nn}^{\lambda} C_n^{\lambda}(\hat{p}_1 \hat{p}_3) \leftarrow \text{orthogonality in } D\text{-space}$$

Remember: $d^D x = \frac{1}{2} S_{D-1} (x^2)^{\frac{D}{2}-1} dx^2 d\hat{x}$, $S_{D-1} = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}$ - surface of the unit hypersphere in R_D

④

$$C_n^\delta(x) = \sum_{p=0}^{[\frac{n}{2}]} \frac{(-1)^p \Gamma(n-p+\delta) (2x)^{n-2p}}{(n-2p)! p! \Gamma(\delta)}$$

$$\frac{(2x)^n}{n!} = \sum_{p=0}^{[\frac{n}{2}]} C_{n-2p}^\delta(x) \frac{(n-2p+\delta) \Gamma(\delta)}{p! \Gamma(n-p+\delta+1)}$$

$$\Rightarrow$$

$$C_n^\delta(x) = \sum_{k=0}^{[\frac{n}{2}]} C_{n-2k}^\delta(x) \frac{(n-2k+\delta) \Gamma(\delta)}{k! \Gamma(\delta)} \frac{\Gamma(n+\delta-k) \Gamma(k+\delta-1)}{\Gamma(n+\delta+1-k) \Gamma(\delta-1)}$$

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If we expand the propagator with the index "1" we will have one-fold series. For the general index "δ" we will have two-fold series.

⑤ It is convenient to work with traceless products: $x^{\mu_1 \dots \mu_n}$

③

$$C_n^>(\hat{x}^{\frac{1}{2}}) = \frac{2^n \Gamma(n+1)}{n! \Gamma(2)} \frac{x^{\mu_1 \dots \mu_n} z^{\mu_1 \dots \mu_n}}{[x^2 z^2]^{\frac{n}{2}}}$$

$$(x^{\mu_1 \mu_2} = x^{\mu_1} x^{\mu_2} - \frac{g^{\mu_1 \mu_2}}{2} x^2 \Rightarrow g_{\mu_1 \mu_2} x^{\mu_1 \mu_2} = 0)$$

The traceless product does not complicate calculations:

$$\int d^D x \frac{x^{\mu_1 \dots \mu_n}}{x^{2\alpha} (x-y)^{2\beta}} = \pi^{\frac{D}{2}} \Gamma(\alpha+\beta) \frac{y^{\mu_1 \dots \mu_n}}{y^{2(\alpha+\beta-\frac{D}{2})}} \quad \text{with } \int d^D x \quad , \quad A^{\mu_1 \dots \mu_n}(\alpha, \beta) = \frac{a_n(\alpha) a_n(\beta)}{a_{n+m}(\alpha+\beta-\frac{D}{2})} , \quad a_n(\alpha) = \frac{\Gamma(\tilde{\alpha}+n)}{\Gamma(\alpha)}$$

!!! But we have Θ -function !!!

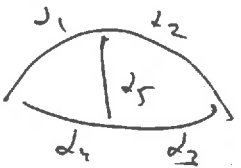
(6)

$$\int d^2x \frac{x^{\mu_1 \dots \mu_4}}{x^{2d} (x-y)^{2\beta}} \partial(x^2 y^2) = \frac{\pi^{\frac{d}{2}} y^{\mu_1 \dots \mu_4}}{y^{2(d+\beta-\frac{d}{2})}} \sum_{n=0}^{\infty} \frac{B(n, 4 | \beta, \frac{d}{2})}{n+d+\beta-\frac{d}{2}} \stackrel{\beta=1}{=} \frac{\pi^{\frac{d}{2}} y^{\mu_1 \dots \mu_4}}{y^{2(\underbrace{d+\beta-\frac{d}{2}}_{d-1})}} \frac{1}{\Gamma(\frac{d}{2})} \frac{1}{(d-1)(n+1)} \quad (d2)$$

$$\int d^2x \frac{x^{\mu_1 \dots \mu_4}}{x^{2d} (x-y)^{2\beta}} \partial(y^2 x^2) = \frac{\pi^{\frac{d}{2}} y^{\mu_1 \dots \mu_4}}{y^{2(d+\beta-\frac{d}{2})}} \sum_{n=0}^{\infty} \frac{B(n, 4 | \beta, \frac{d}{2})}{n+n-d+\frac{d}{2}} \stackrel{\beta=1}{=} \frac{\pi^{\frac{d}{2}} y^{\mu_1 \dots \mu_4}}{y^{2(d-1)}} \frac{1}{\Gamma(\frac{d}{2})} \frac{1}{(n+\frac{d}{2}-1)(n+1)}, \quad (d3)$$

where $B(n, 4 | \beta, \frac{d}{2}) = \frac{\Gamma(n+\beta+n)}{n! \Gamma(n+n+\frac{d}{2}) \Gamma(\beta)} \frac{\Gamma(n+\beta-\frac{d}{2})}{\Gamma(\beta-\frac{d}{2})}$

The sum of (d2) and (d3) $\rightarrow$ (d1) : coming from relations between hypergeometric functions

Using GP method, the results for the diagram  with 3 arbitrary indices have been obtained. They are complicated; they contain ${}_3F_2(\dots|1)$.

(36)

But the result for $I(\alpha)$ is single

$$I_2 = \int \frac{d^d x}{x^2} \frac{1}{x^2} = -2 \frac{\Gamma^2(\lambda) \Gamma(\lambda-\alpha) \Gamma(\alpha+1-2\lambda)}{\Gamma(2\lambda) \Gamma(3\lambda-\alpha-1)} \cdot \left[\frac{\Gamma^2(\frac{\lambda}{2}) \Gamma(3\lambda-\alpha-1) \Gamma(2\lambda-\alpha) \Gamma(\alpha+1-2\lambda)}{\Gamma(\lambda) \Gamma(2\lambda+\frac{\lambda}{2}-\alpha) \Gamma(\frac{3}{2}-2\lambda+\alpha)} \right. \\ \left. + \sum_{n=0}^{\infty} \frac{\Gamma(n+2\lambda)}{\Gamma(n+2\lambda+1)} \frac{1}{n+1+\lambda+\alpha} \right]$$

${}_3F_2(\dots|1)$

Comparing the result with Kazanov one, we see the relation between one ${}_3F_2(\dots|1)$ and two ${}_3F_2(\dots|1-1)$. Now the relation is proven. [A.V.K. and S. Tefer, 2016]

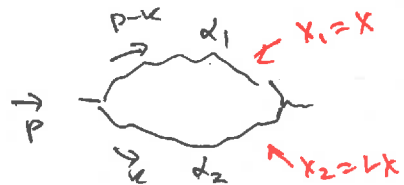
Now I will come to consideration of diagrams with massive propagators. I would like to note the recent paper [D.A. Baikov, K.G. Chetyrkin] where successful expansion of master integrals has been done. [3-loop and 4-loop]

Calculation FD with masses (into propagators)

We can repeat the rules on pages (1-5):

I. Consider one-loop

and definition



$$= \frac{\Gamma(d_1+d_2)}{\Gamma(d_1)\Gamma(d_2)} \int_0^1 dx \cdot x^{d_1-1} (1-x)^{d_2-1} \int \frac{d^D k}{[\underbrace{\quad}]_1^{d_1+d_2}} = (*)$$

$$[\quad]_1 = x[(k-p)^2 + m_1^2] + (1-x)[k^2 + m_2^2] = k^2 + p^2 x - 2(p \cdot k)x + m_1^2 x + m_2^2 (1-x)$$

$$\int \frac{d^D k}{[\quad]_1^{d_1+d_2}} = \pi^{\frac{D}{2}} \frac{\Gamma(d_1+d_2-\frac{D}{2})}{\Gamma(d_1+d_2)} \frac{1}{[m_1^2 x + m_2^2 (1-x) + p^2 x(1-x)]^{d_1+d_2-\frac{D}{2}}}$$

$$(*) = \pi^{\frac{D}{2}} \frac{\Gamma(d_1+d_2-\frac{D}{2})}{\Gamma(d_1)\Gamma(d_2)} \int_0^1 \frac{dx \cdot x^{d_1-1} (1-x)^{d_2-1}}{[m_1^2 x + m_2^2 (1-x) + p^2 x(1-x)]_2^{d_1+d_2-\frac{D}{2}}} = (*)$$

Particular cases

Ia) $m_1 = m_2 = 0$ Consider op fuse 6Ib) $m_2^2 > 0, m_1^2 = -p^2 \equiv m^2$ On-shell

$$[J_2 = m^2 x + p^2 x(L_2) = m^2 x^2] \quad \int_0^1 \frac{dx \cdot x^{\beta-1} (L_2)^{d_2-1}}{\{[J_2 = m^2 x^2]\}^{d_1+d_2-\frac{D}{2}}} = \int_{(L_2)^{d_1+d_2-\frac{D}{2}}} \int_0^1 dx \cdot x^{\beta-1} (L_2)^{d_2-1}$$

$$\beta = d_1 - 2(d_1 + d_2 - \frac{D}{2}) = D - d_1 - 2d_2, \quad \beta + d_2 = D - d_1 - d_2$$

$$\textcircled{*} = \frac{1}{h^{D/2}} \frac{\Gamma(d_1 + d_2 - \frac{D}{2})}{\Gamma(d_1) \Gamma(d_2)} \frac{\cancel{\Gamma(d_2)} \Gamma(D - d_1 - 2d_2)}{\cancel{\Gamma(D - d_1 - d_2)}} \int_{(L_2)^{d_1+d_2-\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}) \Gamma(d_2)}{\Gamma(\beta + d_2)}$$

$c(d_1, d_2)$

II.

$$\begin{aligned}
 & \text{Diagram: A loop with two external lines. The left line has momentum P and index d_1. The right line has index d_2. The loop has two internal lines with masses m_1 and m_2.} \\
 & = \pi^{\frac{d_1+d_2-d_2}{2}} \frac{\Gamma(d_1+d_2-\frac{d_2}{2})}{\Gamma(d_1)\Gamma(\frac{d_2}{2})} \int_0^1 dx \cdot x^{\tilde{d}_1-1} (1-x)^{\tilde{d}_2-1} \\
 & \quad \left[p^2 + \frac{m_1^2}{1-x} + \frac{m_2^2}{x} \right]^{d_1+d_2-\frac{d_2}{2}} \\
 & \quad \text{A red wavy line labeled } \mu^2 \text{ is drawn under the denominator.} \\
 & \quad \text{A green bracket is drawn under the integral.} \\
 & \quad \int_0^1 dx \cdot x^{\tilde{d}_1-1} (1-x)^{\tilde{d}_2-1} \text{ wavy line } \mu^2 \rightarrow p
 \end{aligned}$$

(M3)
One loop can be represented as the integral from a new propagator with effective "mass" dependent on \$x\$

If \$m_1^2=0 \Rightarrow \mu^2 = \frac{m_2^2}{x}\$

If \$m_1^2=m_2^2=m^2 \Rightarrow \mu^2 = \frac{m^2}{x(1-x)}\$

It is convenient to use the inverse mass expansion (no Mellin-Barnes integrals!!!)

[So, we can decrease number of logs in some I=I. !!!]

Example Two loop tadpole with one massless line and two massive lines with $m_1 = m_2 = m$



$$= \frac{\Gamma(d_1, d_2 - \frac{D}{2})}{\Gamma(d_2) \Gamma(d_3)} \cdot \pi^{\frac{D}{2}} \int_0^1 dx \cdot x^{\frac{D}{2}-1} (1-x)^{\frac{D}{2}-1} \cdot \underbrace{\text{Diagram}}_{\pi^{\frac{D}{2}} B(d_1, \beta) \frac{1}{(\mu^2)^{\beta+d_1-\frac{D}{2}}}}$$

$\beta = d_1 + d_2 - \frac{D}{2}$

the integral on x is trivial

$$= \frac{\Gamma(\frac{D}{2} - d_1) \Gamma(d_1 + d_3 - \frac{D}{2}) \Gamma(d_1 + d_2 - \frac{D}{2}) \Gamma(d_2 + d_3 - D)}{\Gamma(\frac{D}{2}) \Gamma(d_2) \Gamma(d_3) \Gamma(2d_1 + d_2 + d_3 - D)}$$

$\Gamma(d_1, d_2, d_3)$

III IBP procedure

(145)

$$\int \frac{d^D k}{(2\pi)^D} \left[\mathcal{D} = \frac{d}{dk_\mu} k_\mu \right] \frac{1}{\{ [k^2 + m_1^2]^{d_1} [(k-p)^2 + m_2^2]^{d_2} [(k-a)^2 + m_3^2]^{d_3} \}} = \int d^D k \left[\frac{d}{dk_\mu} \left(\frac{k_\mu}{\{ \}} \right) - k_\mu \frac{d}{dk_\mu} \frac{1}{\{ \}} \right]$$

= 0
Stokes theorem



$$-k_\mu \frac{d}{dk_\mu} \frac{1}{[k^2 + m_1^2]^{d_1}} = +2d_1 \frac{k^2}{[k^2 + m_1^2]^{d_1+1}} = 2d_1 \left[\frac{1}{[k^2 + m_1^2]^{d_1+1}} - \frac{m_1^2}{[k^2 + m_1^2]^{d_1+1}} \right]$$

$[k^2 + m_1^2]^{d_1}$

$$-k_\mu \frac{d}{dk_\mu} \frac{1}{[(k-p)^2 + m_2^2]^{d_2}} = 2d_2 \frac{(k-p)^2}{[(k-p)^2 + m_2^2]^{d_2+1}} = d_2 \left[\frac{k^2 + m_1^2}{[(k-p)^2 + m_2^2]^{d_2+1}} + \frac{1}{[(k-p)^2 + m_2^2]^{d_2+1}} - \frac{(p^2 + m_1^2 + m_2^2)}{[(k-p)^2 + m_2^2]^{d_2+1}} \right]$$

$[(k-p)^2 + m_2^2]^{d_2}$

Similar

$$-k_\mu \frac{d}{dk_\mu} \frac{1}{[(k-a)^2 + m_3^2]^{d_3}} = d_3 \left[\frac{k^2 + m_1^2}{[(k-a)^2 + m_3^2]^{d_3+1}} + \frac{1}{[(k-a)^2 + m_3^2]^{d_3+1}} - \frac{(p^2 + m_1^2 + m_3^2)}{[(k-a)^2 + m_3^2]^{d_3+1}} \right]$$

$[(k-a)^2 + m_3^2]^{d_3}$

So we have

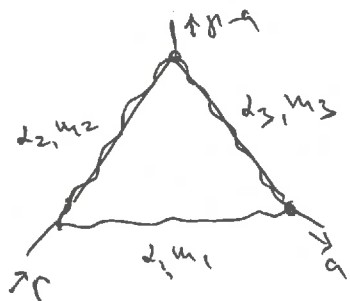
046

$$\underline{D} I + \left[-2d_1 \left\{ \underline{I} - m_1^2 I(d_1 \rightarrow d_1+1) \right\} - 2d_2 \left\{ I \left(\begin{smallmatrix} d_1 \rightarrow d_1-1 \\ d_2 \rightarrow d_2+1 \end{smallmatrix} \right) + \underline{I} - (p^2 + u_1^2 + u_2^2) I(d_2 \rightarrow d_2+1) \right\}_2 - 2d_3 \left\{ d_2 \leftrightarrow d_3 \right\}_2 \right] = 0$$

or

$$(D - 2d_1 - d_2 - d_3) I = d_2 \left\{ I \left(\begin{smallmatrix} d_1 \rightarrow d_1-1 \\ d_2 \rightarrow d_2+1 \end{smallmatrix} \right) - (p^2 + u_1^2 + u_2^2) I(d_2 \rightarrow d_2+1) \right\}_2 + d_3 \left\{ d_3 \leftrightarrow d_2 \right\}_2 - 2d_1 m_1^2 I(d_1 \rightarrow d_1+1)$$

Graphically



$$(D - 2d_1 - d_2 - d_3) = d_2 \left[\begin{array}{c} +1 \\ \text{triangle diagram} \\ -1 \end{array} - (p^2 + \underline{u_1^2 + u_2^2}) \begin{array}{c} +1 \\ \text{triangle diagram} \end{array} \right]$$

$$+ d_3 \left[d_2 \leftrightarrow d_3 \right] - \underline{2d_1 m_1^2} \cdot \begin{array}{c} \text{triangle diagram} \\ +1 \end{array} \quad (\text{IBP})$$

IBP gives a possibility

① to find relations between diagrams and, thus, to work only with some independent set, i.e. with so-called master integrals

② to calculate master integrals using the differential equations, for example.

Example.

The diagram



[A.V. Kotikov, 1991]

1 Step.

$$(\text{bubble with wavy line}) \cdot (D-4) = 2 \left[\text{two-bubble} - \text{cut bubble} - m^2 \cdot \text{cut bubble} \right] \quad (1)$$

$$(\text{bubble with wavy line}) \cdot (D-4) = \text{cut bubble} + \cancel{\text{cut bubble}} - \cancel{\text{cut bubble}} - m^2 \cdot \text{cut bubble} - p^2 \cdot \text{cut bubble} \quad (2)$$

We would like to cancel this diagram
 we can take the following
 combination: $(1) \cdot p^2 - (2) \cdot 2m^2$

$$(1): p^2 - 2m^2(2)$$

$$I(m) \underbrace{(D-4)}_{-2\varepsilon} [p^2 - 2m^2] = 2 \left[p^2 \left\{ \text{diagram 1} - \text{diagram 2} \right\} - m^2 \text{diagram 3} \right. \\ \left. - 2m^2(p^2 - m^2) \underbrace{\text{diagram 4}}_{-\frac{2}{2m^2} I(m)} \right]$$

Diagram 1: Two circles connected by a horizontal line, each with a vertical line through its center. The bottom of each circle is labeled '2'.

Diagram 2: A circle with a horizontal line through its center, labeled '2' at the bottom. The line has a wavy section on the right side.

Diagram 3: A circle with a horizontal line through its center, labeled '2' at the bottom. The line has a wavy section on the right side.

Diagram 4: A circle with a vertical line through its center, labeled '2' at the bottom. The line has a wavy section on the right side.

So, we have the equation for $I(m)$

$$\left[\varepsilon (p^2 - 2m^2) + m^2 (p^2 - m^2) \frac{\partial}{\partial m^2} \right] I(m) = -R$$

$$R = p^2 \left[\underbrace{\text{diagram 1}}_{A(1,1) A(2,1)} - \text{diagram 2} - \frac{m^2}{p^2} \text{diagram 3} \right]$$

Diagram 1: Two circles connected by a horizontal line, each with a vertical line through its center. The bottom of each circle is labeled '2'.

Diagram 2: A circle with a horizontal line through its center, labeled '2' at the bottom. The line has a wavy section on the right side.

Diagram 3: A circle with a horizontal line through its center, labeled '2' at the bottom. The line has a wavy section on the right side.

Diagram 4: A circle with a vertical line through its center, labeled '2' at the bottom. The line has a wavy section on the right side.

contains only simpler (than $I(m)$) diagrams [less number of propagators]

Consider the combination

$$-p^2 \text{ (bubble)} - m^2 \text{ (bubble)} = \textcircled{*}$$

$\uparrow$ IR $\uparrow$ IR

$$\textcircled{*} = (D-4) = 2 \text{ (bubble)} - \text{massless tadpole} + m^2 \text{ (bubble)} - p^2 \text{ (bubble)}$$

different sign

$$\textcircled{*} = -2 \text{ (bubble)} + (D-4) \text{ (bubble)} - 2m^2 \text{ (bubble)} = \textcircled{*}$$

$\uparrow$ IR

$$\textcircled{*} = (D-3) = \text{massless tadpole} - \text{bubble} - m^2 \text{ (bubble)} - 2m^2 \text{ (bubble)}$$

$\uparrow$ IR

$$\textcircled{*} = -2 \text{ (bubble)} + \overbrace{(D-4)}^{-2\epsilon} \text{ (bubble)} + 2 \overbrace{(D-3)}^{1-2\epsilon} \text{ (bubble)} - \text{bubble} + 2m^2 \text{ (bubble)} =$$

$$= 2(1-2\epsilon) \text{ (bubble)} + 4m^2 \text{ (bubble)} = 2 \left(1-2\epsilon - 2m^2 \frac{\partial}{\partial m^2} \right) \text{ (bubble)}$$

(11)

$(2-3\epsilon) = \text{tadpole}_2 \rightarrow \text{massless loop}_2 - \epsilon^2 \text{tadpole}_2$
 $\xrightarrow{\epsilon^2 \frac{\partial}{\partial m^2}} \text{tadpole}_2 + \text{massless loop}_2$

combination of tadpole and massless loop

we should calculate

$\text{tadpole}_2 \approx B(0,2) A(1,1)$, $\text{massless loop}_2 = A(1,1)$

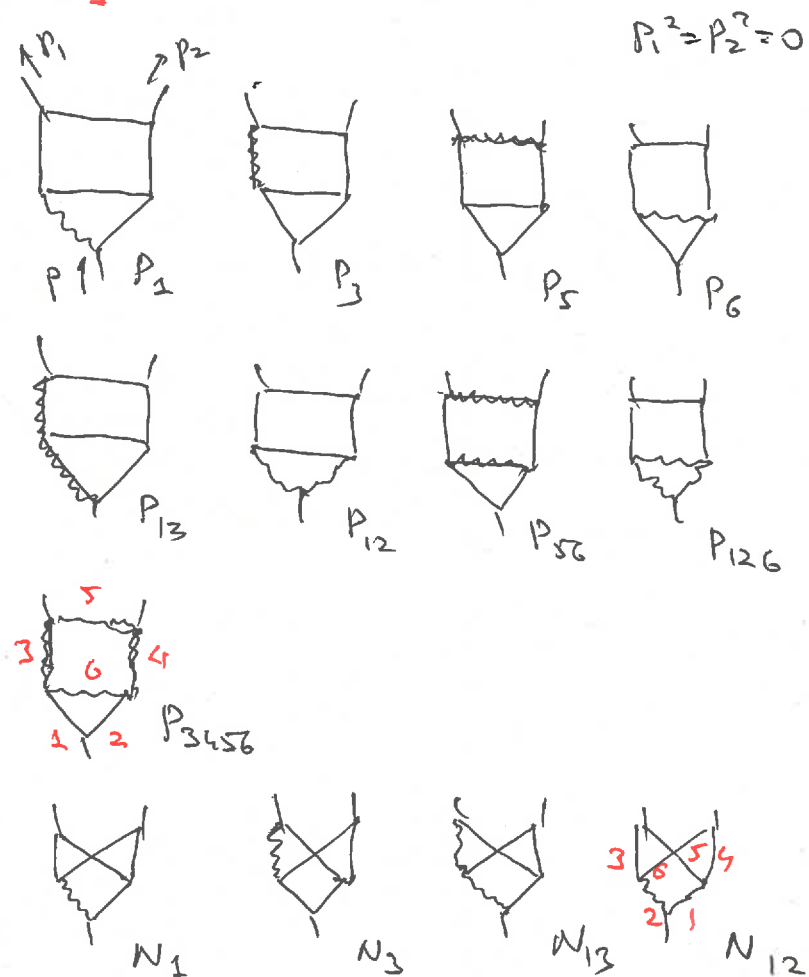
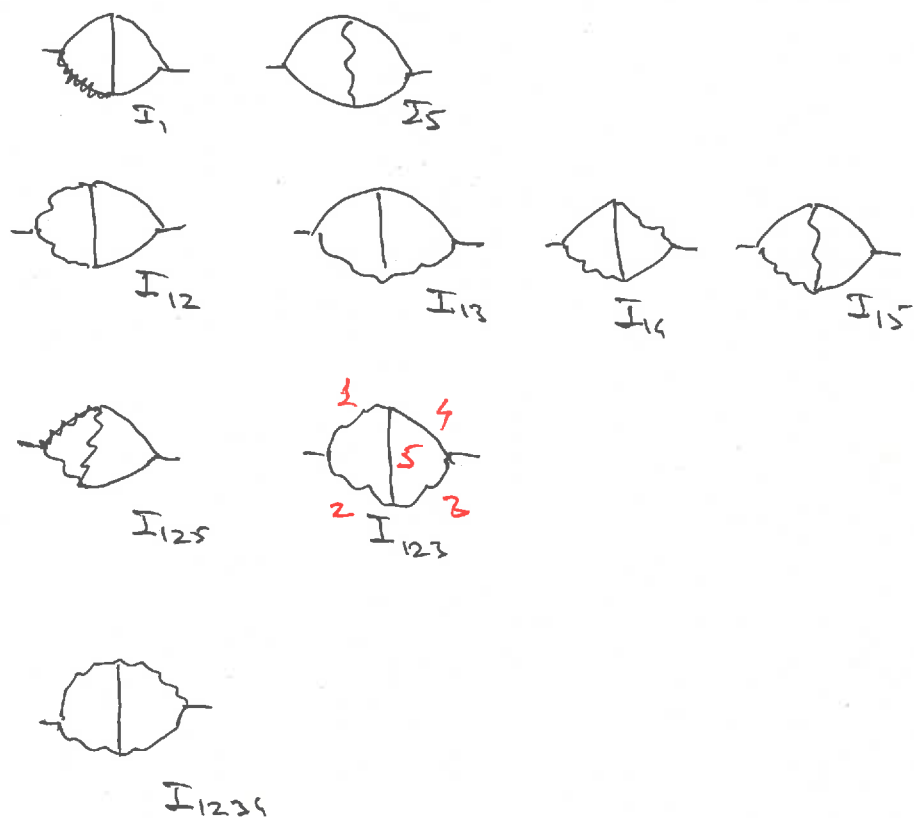
So, we should calculate our loop diagram massless loop_2 (by Feynman parameters, for example) and later to reconstruct the initial diagram $I(m)$ by several integrations

The method is rather powerful but it needs the work with hands (application of computer is problematic)

IV Calculation of ^{some} FI without "direct calculation".

[Y. Fleischer, A.V.K., O.L. Veret'ny, 1999]

(M12)



The results have the form: (inverse mass expansion)

$$FI = \frac{\hat{N}}{p^{2a}} \sum_{n=1}^{\infty} C_n (\gamma x)^n \left\{ F_0^{(n)} + \left[l_n(-x) F_{1,1}(n) + \frac{1}{\varepsilon} F_{1,2}(n) \right] + \right. \\ \left. + \left[l_n^2(-x) F_{2,1}(n) + \frac{1}{\varepsilon} l_n(-x) F_{2,2}(n) + \frac{1}{\varepsilon^2} F_{2,3}(n) + \gamma(z) F_{2,4}(n) \right] + \dots \right\}$$

$$x = \frac{p^2}{m^2}, \gamma = \pm, a=1 \text{ for 2-point FI, } a=2 \text{ for 3-point FI, } \hat{N} = \left(\frac{\bar{\Lambda}^2}{\Lambda^2} \right)^{2\varepsilon}$$

$$C_n = \frac{(h!)^2}{(2n)!} \equiv \hat{C}_n \text{ for FI with two-massive-particle-cuts (2m-cuts)}$$

$$C_n = 1 \text{ for 1m-cuts.}$$

$$F_{n,k}(n) \approx \frac{S_{\pm a, \dots}}{h^b}, \frac{\gamma(\pm a)}{h^b}, \underbrace{\frac{V_{a, \dots}}{h^b}, \frac{W_{a, \dots}}{h^b}}_{\text{only for 2m-cuts}}$$

$$S_{\pm a, \pm b, \dots}^{(j)} = \sum_{m=1}^j \frac{(\pm)^m}{h^a} S_{\pm b, \dots}(m), \quad \gamma(\pm a, \pm b, \dots) = \sum_{m=1}^{\infty} \frac{(\pm)^m}{h^a} S_{\pm b, \dots}(m)$$

$$V_{a, b, \dots}^{(j)} = \sum_{m=1}^j \frac{\hat{C}_m}{m} S_{b, \dots}(m), \quad W_{a, b, \dots}^{(j)} = \sum_{m=1}^j \frac{\hat{C}_m}{m} S_{b, \dots}(m)$$

$$\text{let } \phi_{\pm a_1, \dots, \pm a_n} \sim \delta(\pm a_1, \dots, \pm a_n) \sim \frac{1}{n!} \left(\sum_i a_i = a \right)$$

(M14)

↑ transcendental weight
(level of complexity)

Then

$$F_{N,k}(n) \sim \frac{1}{n^{3-N}} \quad (N \geq 2) \quad \text{for two-point FI}$$

$$F_{N,k}(n) \sim \frac{1}{n^{4-N}} \quad (N \geq 3) \quad \text{for three-point FI}$$

In x^k space the corresponding property for weights of

Polylogarithms

(with argument x , when $C_k = 1$
and y , when $C_k = \hat{C}_k$)

So, if we calculated the expansion of some poles (or the coefficient in the front of logs) we can predict more

$$y = \frac{1 - \sqrt{x/x-4}}{1 + \sqrt{x/x-4}}$$

complicated terms by increasing of level of complexity.

The number of terms ("basis") is limited.

Using our calculation of the first coefficient ($n \leq 100$)

we can predict exact result (at any n).

Moreover, if we calculated (in some way) two-point diagrams we can predict the results for ~~calculation~~ three-point FIS.

(115)

Indeed,

$$I_2 = \frac{\hat{N}}{q^2} \sum_{n=1}^{\infty} \frac{x^n}{n} \left\{ \frac{1}{2} \ln^2(-x) - \frac{2}{n} \ln(-x) + g_2 - 2g_2 - 2 \frac{S_1}{n} + \frac{3}{n^2} \right\}$$

$$P_1 = \frac{\hat{N}}{(q^2)^2} \sum_{n=1}^{\infty} \frac{(-x)^n}{n} \left\{ -\frac{1}{2g^3} - \frac{1}{2g^2} S_1 + \frac{1}{2} \left[-\frac{1}{2} g_2 + \frac{5}{2} S_2 - \frac{1}{2} S_1^2 + \frac{1}{n^2} - \frac{1}{n} \ln(-x) + \frac{1}{2} \ln^2(-x) \right] \right.$$

$$- \frac{8}{3} g_3 - g_2 S_1 - \frac{g_2}{n} + \frac{8}{3} S_3 + \frac{9}{2} S_1 S_2 + \frac{5}{6} S_1^3 + 4 \frac{S_2}{n} + 2 \frac{S_1}{n^2} + \frac{3}{n^3} +$$

$$\left. + (g_2 - 4S_2 - 2 \frac{S_1}{n} - \frac{3}{n^2}) \ln(-x) + (S_1 + \frac{1}{2n}) \ln^2(-x) - \frac{1}{2} \ln^3(-x) \right\}$$

$$I_5 = \frac{\hat{N}}{q^2} \sum_{n=1}^{\infty} \frac{(-x)^n}{n} \left\{ -\ln^2(-x) + \frac{2}{n} \ln(-x) - 2g_2 - 4S_2 - \frac{2}{n^2} - \frac{2(-1)^n}{n^2} \right\}$$

$$P_5 = \frac{\hat{N}}{(q^2)^2} \sum_{n=1}^{\infty} \frac{(-x)^n}{n} \left\{ -6g_3 + 2g_2 S_1 + 6S_3 - 2S_1 S_2 + 4 \frac{S_2}{n} - \frac{S_1^2}{n} + 2 \frac{S_1}{n} + \right.$$

$$\left. + (-4S_2 + S_1^2 - 2 \frac{S_1}{n}) \ln(-x) + S_1 \ln^2(-x) \right\}$$

$$I_{12} = \frac{N}{q^2} \sum_{h=1}^{\infty} \frac{x^h}{h^2} \left\{ \frac{1}{h} + \hat{C}_h \left(-2\ln(-x) - 3W_{4,1} + \frac{2}{h} \right) \right\}$$

$$P_{12} = \frac{N}{(q^2)^2} \sum_{h=1}^{\infty} \frac{x^h}{h^2} \hat{C}_h \left[\frac{2}{\varepsilon^2} + \frac{2}{\varepsilon} \left(S_1 - 3W_1 + \frac{1}{h} - \ln(-x) \right) + 12W_2 - 18W_{4,1} - 13S_2 + S_1^2 - 6S_1W_1 \right. \\ \left. + \frac{2S_1}{h} + \frac{2}{h^2} - 2 \left(S_1 + \frac{1}{h} \right) \ln(-x) + \ln^2(-x) \right]$$

$$W_{12} = \frac{N}{(q^2)^2} \sum_{h=1}^{\infty} \frac{x^h}{h} \hat{C}_h \left[-\frac{6}{\varepsilon^2} W_1 + \frac{1}{\varepsilon} \left[-4W_2 - 12W_{4,1} - 12S_1W_1 - 16S_2 \right] - \right. \\ \left. - 12S_2W_1 - 4W_3 - 8S_3 + 12W_{1,2} - 8W_{2,1} - 12W_{4,(1+1)} \right]$$

(i.e. no mixture of function with different transcendental weight)

in Fw

special form

Same level of complexity $\checkmark$ corresponds to the $\checkmark$ DE for FI (A.V.K, 2010)
(at least for $\checkmark$ planar case):

$$\left(x \frac{d}{dx} + K\varepsilon \right) FI = \text{less complicated diagrams} (\equiv FI_1)$$

For the FI_1 we can prepared ~~same~~ ^{similar} equation with
r.h.s, containing only less² complicated diagrams ($\equiv FI_2$)

Now, it is very popular the method to use the matrix homog.

(M17)

equation in the so-called ε -form (J. Henn, 2013)

$$\frac{d}{dx} \hat{F}I - \varepsilon \hat{K} \hat{F}I = 0, \quad \text{where}$$

$$\hat{F}I = \begin{pmatrix} FI \\ FI_2/\varepsilon \\ \vdots \\ FI_n/\varepsilon^n \end{pmatrix}$$

!!! [lot of application] !!!
in the last years

V Elliptic results

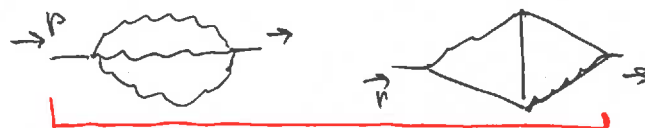
[Remiddi et al., 2005, 2016, 2017]

[B.A. Kucel et al., 2006]

[K. Adams et al., 2014-2018]

[S. Bloch et al., 2013, 2016]

(1418)



Simplest 3m-cuts diagrams with elliptics in their results

[three-massive-particle-cuts (3m-cuts)]

We consider



$q^2 = m^2$
Min. space

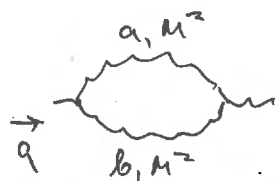
, which is important for non-relativistic QED and QCD, when $M=m$

$$\text{Here } \longrightarrow = \frac{1}{p^2 - m^2}$$

$$\text{~~~~~} = \frac{1}{p^2 - M^2}$$

Min. space !!

Effective mass method:



$$= i^{1+d} \frac{\Gamma(a+b-\frac{D}{2})}{\Gamma(a)\Gamma(b)}$$

$$\int_0^1 \frac{ds}{(1-s)^{a+1-\frac{D}{2}} s^{b+1-\frac{D}{2}}}$$

$$\frac{a+b-\frac{D}{2}}{\mu^2 = \frac{M^2}{s(1-s)}}$$

M19

Applying DE method to one loop diagrams with the effective mass and ~~more~~ integrated later on s we have $(x = \frac{m^2}{M^2})$

$$M^2 \Gamma_{122}^{MM} = \frac{1}{x} \sum_{h=1}^{\infty} (-x)^h \bar{C}_h \cdot \frac{1}{2h} \left[-\ln x + S_1 - 3\bar{S}_1 + 2\bar{\bar{S}}_1 + \frac{1}{2h^2} \right] +$$

$$+ \frac{1}{x^2} \sum_{h=1}^{\infty} \frac{(-16x^2)^h}{\binom{2h}{h} \binom{4h}{2h}} \cdot \frac{1}{2} \left[-\frac{1}{2h-1} + \frac{1}{2h} - \frac{1}{4h^2} \right]$$

$$\bar{C}_h = \frac{\binom{2h}{h}}{\binom{4h}{2h}}$$

$$S_1 = S_1(h-1)$$

$$\bar{S}_1 = S_1(2h-1)$$

$$\bar{\bar{S}}_1 = S_1(4h-1)$$

$$(M^2)^{2\epsilon} \Gamma_{112}^{MM} = -\frac{1}{2\epsilon^2} - \frac{1}{\epsilon} \left(\ln x + \frac{1}{2} \right) - \ln^2 x - \ln x - \frac{1}{2} \eta_2 - \frac{1}{\epsilon}$$

$$+ \frac{1}{x} \sum_{h=1}^{\infty} (-x^2)^h \bar{C}_h \cdot \frac{4h-1}{2h(2h-1)^2} \left[\ln x - S_1 + 3\bar{S}_1 - 2\bar{\bar{S}}_1 - \frac{1}{2h} - \frac{2}{2h-1} + \frac{2}{4h-1} \right]$$

$$= \sum_{h=1}^{\infty} \frac{(-16x^2)^h}{\binom{2h}{h} \binom{4h}{h}} \frac{1}{4h^2(2h+1)}$$

Γ_{111}^{MM} has similar structure

Integral representations

$$M^2 J_{122}^{MMM} = -\frac{1}{2x} \int_0^1 \frac{dt}{t\sqrt{1-t}} \left(\frac{1}{\sqrt{1+4A^2}} L(A) - 2\ln A \right)$$

$$A = \frac{x^2}{4}, \quad L(x) = L_1(x) + L_2(x)$$

$$L_1(x) = \ln \frac{\sqrt{x+1}}{\sqrt{x-1}}, \quad L_2(x) = \frac{\sqrt{-2x}}{\sqrt{x+2x}}$$

$$(M^2)^{2c} J_{112}^{MMM} = -\frac{1}{2x^2} - \frac{1}{2} \left(\ln x - \frac{1}{2} \right) - \ln^2 x - \ln x - \frac{1}{2} \eta(2) - \frac{1}{2}$$

$$-\frac{1}{x} \int_0^1 \frac{dt}{t^2 \sqrt{1-t}} \left(\sqrt{1+4x^2} L(A) - 2\ln A + 4A \right)$$

Integral J_{111}^{MMM} has similar results

$\int_0^1 \frac{dt}{t\sqrt{1-t}\sqrt{1+4A^2}}$ is the Elliptic integral of the third kind [EIK]

So, the results of above integrals are integrals of EIK.

They should be expressed in the form of Elliptic Polylogarithms.

Now we are working for this subject.

Conclusion

I have considered several approaches to calculate Feynman integrals (some of them are very old)

- Feynman parameterization
- Integration by parts
- Method of Uniqueness + transformation
- Gegenbauer Polynomials
- Differential equations
- method of "effective mass".

Thank you for your attentions !!!