Some methods for calculations of Feynman diagrams

A.V. Kotikov (JINR, Dubna)

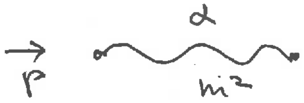
Main reference: D.I Kazakov,
"Analytical methods for Multiloop calculations: Two Lectures on the Method of Uniquness",
JINR preprint JINR-E2-84-479
Dubna (1984)

Ⓘ Definition: Euclidean space, $D = 4-2\varepsilon$, $\lambda = D/2 - 1 = 1-\varepsilon$

dimensional regularization

$$d^4k \to d^Dk\,(\mu^2)^{\varepsilon}$$

$$\equiv \frac{1}{[p^2]^{\alpha}} \equiv \frac{1}{p^{2\alpha}}$$



$$\equiv \frac{1}{[p^2+m^2]^{\alpha}}$$

Ⅱ Tadpoles

1. 

$$\equiv \int \frac{d^Dk}{k^{2\beta}\,[k^2+m^2]^\alpha} = \otimes$$

polar. coordinates

$$d^Dk = \frac{1}{2}(k^2)^{\overbrace{D/2-1}^{\lambda}}\, dk^2\, d\Omega$$

$$\int d\Omega = S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$ is the surface of the unit hypersphere in R^D

↑ Euler Γ-function

$$\otimes = \left[k^2 = m^2 z\right] = \frac{S_D}{2}\underbrace{\int_0^\infty \frac{dz\cdot z^{\lambda-\beta}}{(z+1)^\alpha}}\ \frac{1}{(m^2)^{\alpha+\beta-D/2}} = \circledast$$

Euler $\hat{B}$-function : $\hat{B}(\beta,\alpha) = \frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\beta+\alpha)}$

$$\int_0^\infty \frac{dz\cdot z^\gamma}{(1+z)^\alpha} = \int_1^\infty \frac{dt\cdot(t-1)^\gamma}{t^\alpha} = \underbrace{\int_0^1 dp\ p^{\alpha-\gamma-2}(1-p)^\gamma}_{\text{Standard form}} = \hat{B}(\alpha-\gamma-1,\gamma+1)$$

$z+1=t$, $t=\frac{1}{p}$

$$\circledast = \underbrace{\boxed{\frac{\pi^{D/2}}{\Gamma(D/2)}\ \frac{\Gamma(D/2-\beta)\,\Gamma\left(\overset{\alpha+\beta-D/2}{\overline{\alpha+\beta-D/2}}\right)}{\Gamma(\alpha)}}}_{B(\beta,\alpha)}\ \frac{1}{(m^2)^{\alpha+\beta-D/2}}$$

Ⓐ

2. If $\alpha+\beta < D/2$,

$\xrightarrow[m^2\to 0]{} 0$

(A1)

Dimension regularization (convension):

$\underset{m^2=0}{\equiv 0}$ for any α and β

in real calculation

$[\alpha+\beta=D/2$, δ-function; $\alpha\to\alpha+\delta]$

$\delta(\beta+\alpha-D/2)$

S.G. Gorishnii, A.P. Isaev
Theor. Math. Phys. 62 (1985) 232

3. Consider the integral

$$\int \frac{d^D k}{[k^2 - 2pk + m^2]^\alpha} = \circledast$$

Introduce $k_1 = k - p \Rightarrow [\ \ ] = k_1^2 + \underbrace{m^2 - p^2}_{m_1^2}$

Moreover, $d^D k = d^D k_1$

$$\circledast = \int \frac{d^D k_1}{[k_1^2 + \underbrace{m^2 - p^2}_{m_1^2}]^\alpha} = \pi^{D/2} B(0,\alpha) \frac{1}{[\underbrace{m^2 - p^2}_{m_1^2}]^{\alpha - D/2}} \qquad (A2)$$

↑ "effective mass" (very important for future calculations)

III Feynman parametrization (here $[k^2+m^2] \equiv [k]$)

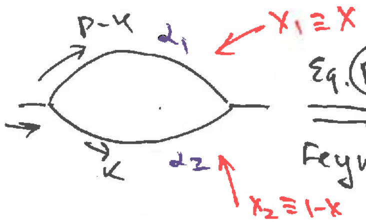
$$\frac{1}{\prod_{i=1}^{N}[k_i]^{\alpha_i}} = \frac{\Gamma\left(\sum_i^N \alpha_i\right)}{\prod_1^N \Gamma(\alpha_i)} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \dots \int_0^{1-\sum_1^{N-2} x_i} dx_{N-1} \cdot \frac{\prod_{i=1}^{N} x_i^{\alpha_i-1}}{\left(\sum_{i=1}^{N} x_i[k_i]\right)^{\sum_1^N \alpha_i}} \Bigg|_{x_N = 1-\sum_1^{N-1} x_i} \qquad (B)$$

when $N=2$

$$\frac{1}{[k_1]^{\alpha_1}[k_2]^{\alpha_2}} = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 dx_1 \frac{x_1^{\alpha_2-1}(1-x_1)^{\alpha_2-1}}{\left(x_1[k_1]+(1-x_1)[k_2]\right)^{\alpha_1+\alpha_2}} \qquad (B1)$$

Feynman parametrization transform products of propagators to the new propagator with the effective mass, depending on x_i. For this new propagator, we can apply our formulas for tadpoles.

IV One loop (massless loop)

1. 

$$\underset{\text{Feyn. par.}}{\overset{\text{Eq.}\,(B1)}{=\!=}} \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 dx \cdot x^{\alpha_1-1}(1-x)^{\alpha_2-1} \int \frac{d^D k}{[\;]_1^{\alpha_1+\alpha_2}} = ⊛$$

$$[\;]_1 = x(k-p)^2 + (1-x)k^2 = k^2 + p^2x - 2(pk)x$$ ← Tadpole with $p \to p_1 = px$.
$m^2 = p^2x$

$$\int \frac{d^D k}{[\;]_1^{\alpha_1+\alpha_2}} \overset{\text{Eq. (A2)}}{=\!=} \pi^{\frac{D}{2}} \frac{\Gamma(\alpha_1+\alpha_2-\frac{D}{2})}{\Gamma(\alpha_1+\alpha_2)} \frac{1}{[\underbrace{\underbrace{p^2x}_{m^2} - \underbrace{p^2x^2}_{p_1^2}}_{m_1^2}]_2}$$

$$[\;]_2 = p^2x(1-x)$$

$$\boxed{\tilde{\alpha}_i = \frac{D}{2} - \alpha_i}$$

$$⊛ = \pi^{\frac{D}{2}} \frac{\Gamma(\alpha_1+\alpha_2-\frac{D}{2})}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{1}{(p^2)^{\alpha_1+\alpha_2-\frac{D}{2}}} \underbrace{\int_0^1 \frac{dx \cdot x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{[x(1-x)]^{\alpha_1+\alpha_2-\frac{D}{2}}}}_{} =$$

$$\int_0^1 dx \cdot x^{\tilde{\alpha}_2-1}(1-x)^{\tilde{\alpha}_1-1} = \frac{\Gamma(\tilde{\alpha}_2)\Gamma(\tilde{\alpha}_1)}{\Gamma(\tilde{\alpha}_1+\tilde{\alpha}_2)}$$

$$= \pi^{\frac{D}{2}} \underset{\substack{\uparrow \\ G}}{A(\alpha_1,\alpha_2)} \frac{1}{(p^2)^{\alpha_1+\alpha_2-\frac{D}{2}}} \quad (\bigstar)$$

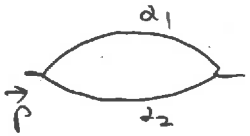
$$\boxed{A(\alpha_1,\alpha_2) = \frac{a(\alpha_1)\,a(\alpha_2)}{a(\alpha_1+\alpha_2-\frac{D}{2})}, \quad a(\alpha) = \frac{\Gamma(\tilde{\alpha})}{\Gamma(\alpha)}}$$

2. Recover Eq. (c) using Fourier transforms

$$\int \frac{d^D p\, e^{ipx}}{p^{2\alpha}} = \frac{\pi^{D/2}\, 2^{\overset{2\tilde{\alpha}}{\overline{D-2\alpha}}}\, a(\alpha)}{x^{2\tilde{\alpha}}}$$

$$\int \frac{d^D x\, e^{-ipx}}{x^{2\alpha}} = \frac{\pi^{D/2}\, 2^{2\tilde{\alpha}}\, a(\alpha)}{p^{2\tilde{\alpha}}} \Rightarrow \frac{1}{p^{2\alpha}} = \frac{1 \quad a(\alpha)}{\pi^{D/2}\, 2^{2\alpha}\, \cancel{a(\tilde{\alpha})}} \int \frac{d^D x\, e^{-ipx}}{x^{2\tilde{\alpha}}}$$

Then



$$= \int \frac{d^D k}{k^{2\alpha_1}(p-k)^{2\alpha_2}} = \boxed{\int d^D k}\ \frac{1}{\pi}\ \frac{a(\alpha_1)a(\alpha_2)}{2^{2(\alpha_1+\alpha_2)}} \int \frac{d^D x_1 \boxed{e^{-ikx_1}}}{x_1^{2\tilde{\alpha}_1}} \int \frac{d^D x_2 \boxed{e^{-i(p-k)x_2}}}{x_2^{2\tilde{\alpha}_2}} =$$

$$\int d^D k\, e^{ik(x_2-x_1)} = (2\pi)^D \delta^D(x_2-x_1)$$

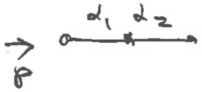
$$= \frac{\cancel{2^D}\, a(\alpha_1)a(\alpha_2)}{\cancel{2^{2(\alpha_1+\alpha_2)}}} \underbrace{\int \frac{d^D x_1\, e^{-ipx_1}}{x_1^{2(\tilde{\alpha}_1+\tilde{\alpha}_2)}}}_{\frac{\pi^{D/2}\, \cancel{2^{D-2(\tilde{\alpha}_1+\tilde{\alpha}_2)}}\, a(\tilde{\alpha}_1+\tilde{\alpha}_2)}{(p^2)^{-(\tilde{\alpha}_1+\tilde{\alpha}_2)+D/2}}} = \frac{\pi^{D/2} a(\alpha_1)a(\alpha_2)}{a(\alpha_1+\alpha_2-D/2)}\ \frac{1}{(p^2)^{\alpha_1+\alpha_2-D/2}}$$

$\boxed{\text{Ok}}$

$\tilde{\alpha}_1+\tilde{\alpha}_2 = D-\alpha_1-\alpha_2$

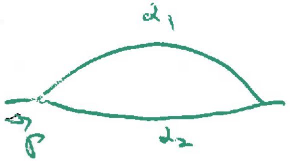
$D/2-(\tilde{\alpha}_1+\tilde{\alpha}_2) = \alpha_1+\alpha_2-D/2$

3. Chain

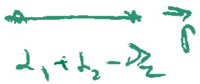


$$= \frac{1}{(p^2)^{\alpha_1}} \frac{1}{(p^2)^{\alpha_2}} = \frac{1}{(p^2)^{\alpha_1+\alpha_2}} =$$

(C1)



$$= \pi A(\alpha_1,\alpha_2)$$



(C)

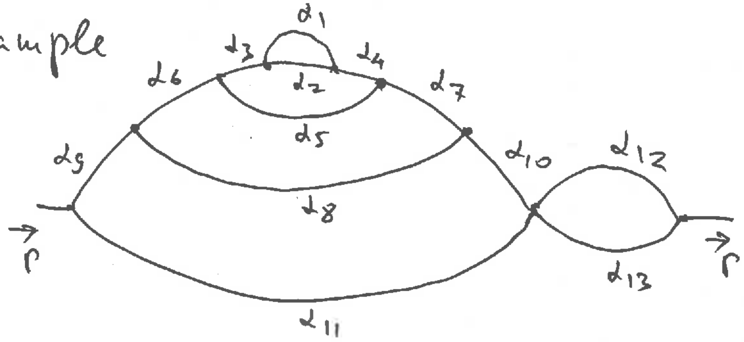
$$A(\alpha_1,\alpha_2) = \frac{a(\alpha_1)a(\alpha_2)}{a(\alpha_1+\alpha_2-D/2)}$$

$=$

(C1)

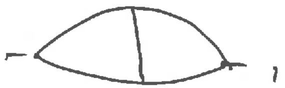
In many cases Eqs. (C) and (C1) give a possibility to evaluate Feynman integrals without any direct calculations of some integrals

4. Example



= ⊛

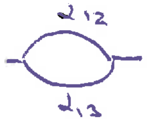
But ~~at the~~ starting at two-loop there are diagrams like



, which is not a combination of loops and chains

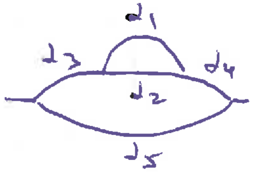
Steps

①

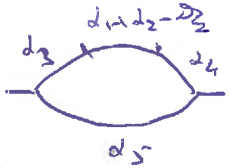


$= \pi^{D/2} A(\alpha_{12},\alpha_{13}) \xrightarrow{\alpha_{12}+\alpha_{13}-D/2}$

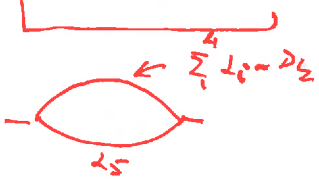
②

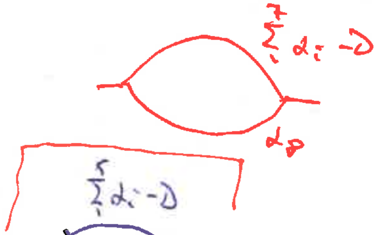


$= \pi^{D/2} A(\alpha_1,\alpha_2)$

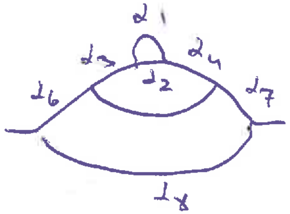


$= \pi^{D} A(\alpha_1,\alpha_2)\, A(\sum_1^4 \alpha_i - D/2, \alpha_5) \xrightarrow{\sum_1^5 \alpha_i - D}$





③



$= \pi^{D} A(\alpha_1,\alpha_2)\, A(\sum_1^4 \alpha_i - \frac{D}{2}, \alpha_5)$

$= \pi^{\frac{3D}{2}} A(\alpha_1,\alpha_2)\, A(\sum_1^4 \alpha_i - \frac{D}{2}, \alpha_5)\, A(\sum_1^7 \alpha_i - D, \alpha_8) \xrightarrow{\sum_1^8 \alpha_i - \frac{3D}{2}}$

$$⊛ = \pi^{\frac{5D}{2}} A(\alpha_1,\alpha_2)\, A(\sum_1^4 \alpha_i - \frac{D}{2}, \alpha_5)\, A(\sum_1^7 \alpha_i - D, \alpha_8)\, A(\sum_1^{10} \alpha_i - \frac{3D}{2}, \alpha_{11}) \cdot A(\alpha_{12},\alpha_{13}) \xrightarrow{\sum_1^{13} \alpha_i - \frac{5D}{2}}$$

number of loops

5. To find the real result ("numbers") it is necessary to use the relation $\Gamma(x+1)=x\Gamma(x)$ to transform of Γ-functions to the form $\Gamma(1+a\varepsilon)$ and later to use

$$\Gamma(1+a\varepsilon)=\exp\left[-\gamma_E a\varepsilon+\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}(-1)^k a^k\varepsilon^k\right]$$

number of integration

Note: the term $\pi^{D/2\cdot k}$ can be not collected exactly, because any integration gives:

$$\frac{\pi^{D/2}\,\Gamma(1+\varepsilon)}{(2\pi)^{D}}\cdot g^2(\mu^2)^\varepsilon = \frac{g^2\,\Gamma(1+\varepsilon)}{(2\pi)^{D/2}}(\mu^2)^\varepsilon = \underbrace{\frac{g^2}{16\pi^2}}_{\alpha}\underbrace{(4\pi)^\varepsilon\Gamma(1+\varepsilon)(\mu^2)^\varepsilon}_{(\bar{\mu}^2)^\varepsilon}$$

from measure of FI

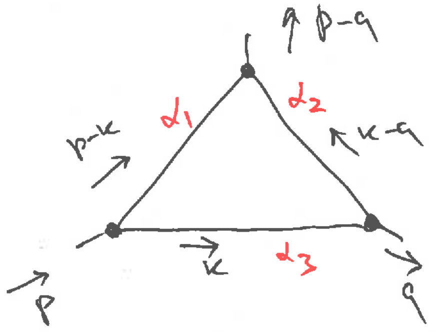
charge

dim.reg.mass parameter

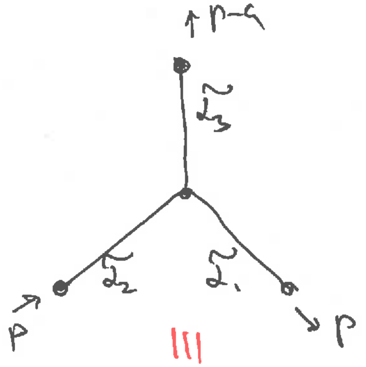
coupling constant

new dim. scale ≡ ≡ $\overline{MS}$-scheme

V Uniqueness relation: [M. D'Eramo, L. Peliti, G. Parisi, 1971] [A.N. Vassiliev, Yu. M. Pis'mak, Yu. R. Honkonen, 1981]



$$\boxed{\sum \alpha_i = D} \quad \equiv \quad \pi^{D/2} A(\alpha_1, \alpha_2)$$



[?! Dissertation of A.M. Polyakov, 19?? ?!]

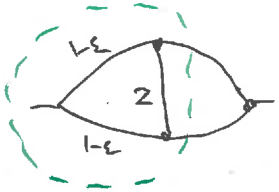
$$\int \frac{d^Dk}{k^{2\alpha_3}[(p-k)^2]^{\alpha_1}[(k-q)^2]^{\alpha_2}} = ⊛ \qquad ||| \qquad \frac{1}{[(p-q)^2]^{\tilde{\alpha}_3}\, p^{2\tilde{\alpha}_2}\, q^{2\tilde{\alpha}_1}}$$

To prove: inversion $\left(k \to \frac{1}{k_1},\ p \to \frac{1}{p_1},\ \frac{1}{q_1}\right) \subset$ conformal transformation (angles are kept $\equiv$ scalar products)

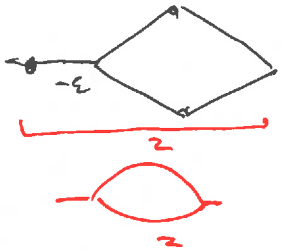
$$k^2 \to \frac{1}{k_1^2},\ (p-k)^2 = p^2 - 2(pk) + k^2 \to \frac{1}{p_1^2} - \frac{2(p_1k_1)}{p_1^2k_1^2} + \frac{1}{k_1^2} = \frac{(p_1-k_1)^2}{p_1^2k_1^2}, \quad d^Dk = \frac{1}{2}(k^2)^{\frac{D}{2}-1} dk^2\, d\Omega \to \frac{d^Dk_1}{(k_1^2)^D}$$

$$⊛ = \underbrace{\int \frac{d^Dk_1\, \cancel{k_1^{2\alpha_3}}\, (\cancel{k_1^2}\, p_1^2)^{\alpha_1} (\cancel{k_1^2}\, q_1^2)^{\alpha_2}}{\cancel{(k_1^2)^{D}}\, [(p_1-k_1^2]^{\alpha_1}\, [(k_1-q_1)^2]^{\alpha_2}}}_{\text{loop in the new variables}}$$

Examples

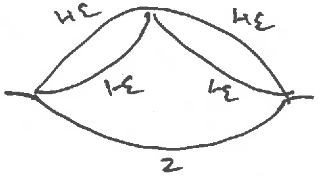


$= \pi^{D/2} A(1-\varepsilon, 1-\varepsilon)$

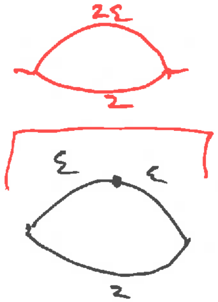


$= \pi A(1-\varepsilon,1-\varepsilon)\, A(2,2)$

$= \frac{1}{\pi^{D/2}} \frac{1}{A(1-\varepsilon,1-\varepsilon)}$



$= \frac{\pi^{D/2} A^2(1-\varepsilon,1+\varepsilon)}{A(1-\varepsilon,-\varepsilon)}$



$=$

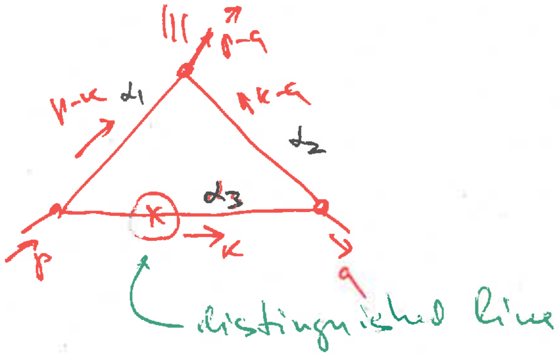
$= \pi^{D} \frac{A^2(1-\varepsilon,1+\varepsilon)}{A(1-\varepsilon,1-\varepsilon)} A(2,2\varepsilon)$

$\underline{\text{VI}}$ IBP (integration by parts) procedure

[K.G. Chetyrkin, F.V. Tkachov, 1981] → p-space
[A.N. Vassiliev, et al. 1981] → x-space

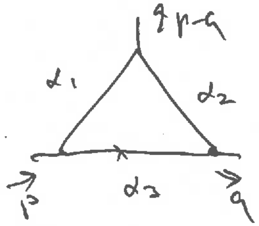
$$\int \frac{d^D k \quad \left[D = \frac{d}{dk_\mu} k_\mu\right]}{\left[k^{2\alpha_3}(k-q)^{2\alpha_2}(k-p)^{2\alpha_1}\right]_+} = \int d^D k \left[\underbrace{\frac{d}{dk_\mu}\left(\frac{k^\mu}{[\,]_+}\right)}_{=0} - k^\mu \frac{d}{dk^\mu}\frac{1}{[\,]_+} \right]$$

Stoks theorem

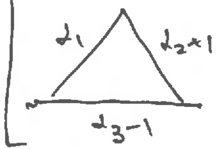


$$-k^\mu \frac{d}{dk^\mu}\frac{1}{k^{2\alpha_3}} = \frac{2\alpha_3}{k^{2\alpha_3}}$$

$$-k^\mu \frac{d}{dk^\mu}\frac{1}{(k-q)^{2\alpha_2}} = 2\alpha_2 \frac{(k,k-q)}{(k-q)^{2(\alpha_2+1)}} = \alpha_2 \frac{k^2+(k-q)^2-q^2}{(k-q)^{2(\alpha_2+1)}}$$

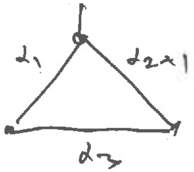


$(D-2\alpha_3-\alpha_2-\alpha_1) = \alpha_2 \Big[$

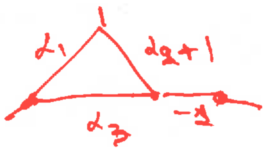


$-$

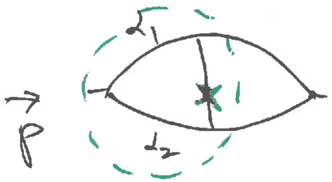
$-q^2$



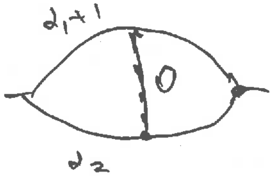
$\Big] + \alpha_1 \Big[\alpha_1 \leftrightarrow \alpha_2,\ p \leftrightarrow q \Big]$



Example:



$(D-2-\alpha_1-\alpha_2) = \alpha_1 \Big[$



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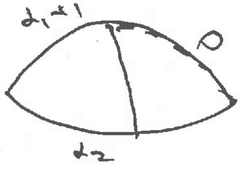
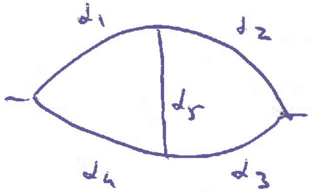


$\Big] + \alpha_2 \big[\alpha_1 \leftrightarrow \alpha_2 \big] =$

$$= \Bigg\{ \Big[\alpha_1 \overbrace{A(\cancel{\;},1)}^{A(1,1)} \Big(A(\alpha_1+1,\alpha_2) - A(\alpha_1+1,\alpha_2+\varepsilon \Big) \Big] + \big[\alpha_1 \leftrightarrow \alpha_2 \big] \Bigg\} \frac{1}{(p^2)^{\alpha_1+\alpha_2-1+2\varepsilon}}$$

VII Transformations.

Relations between FI

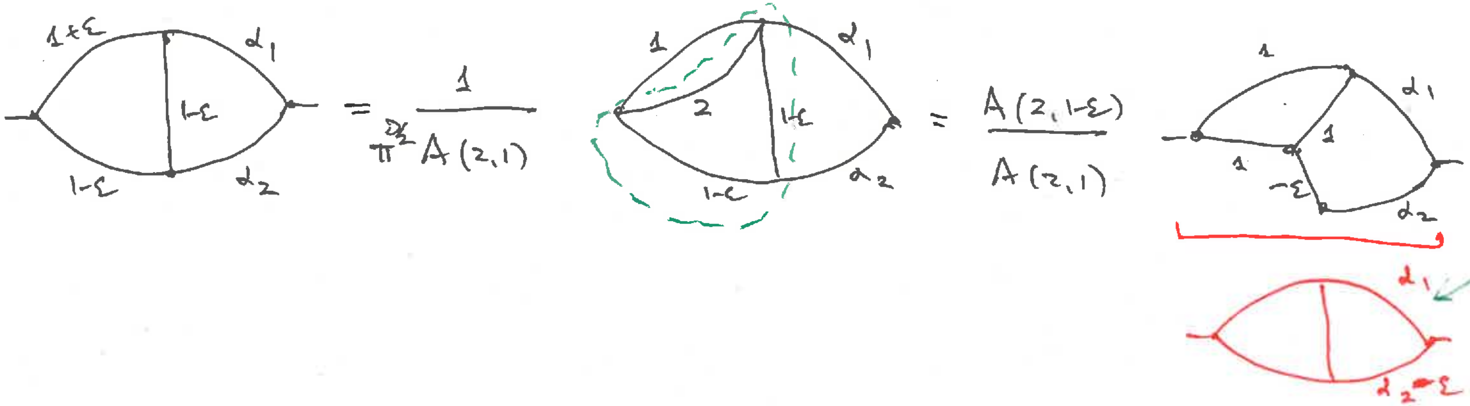


with different α_i.

[A.N. Vassiliev et. al., 1981]
[S.G. Gorishnii, A.P. Isaev, 1985]
[D.T. Barfoot, D.J. Broadhurst, 1988] → very big group of transformations

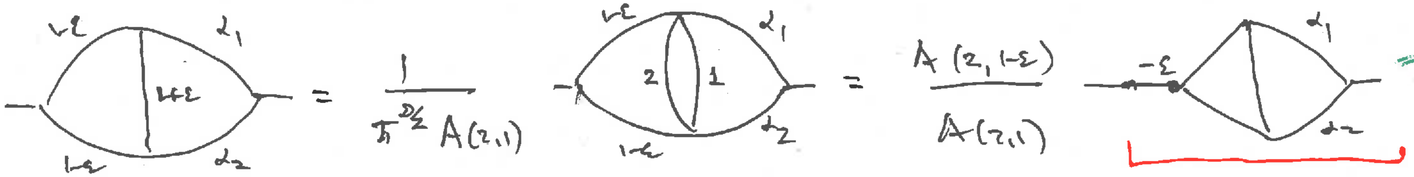
Examples

(I) The additional loop → to "lateral" line



Calculated before by IBP

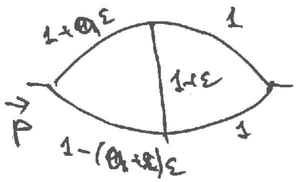
(II) The additional loop → to vertical line



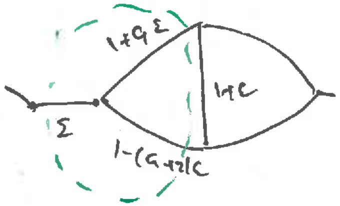
→ Calculated before by IBP

III Additional propagator

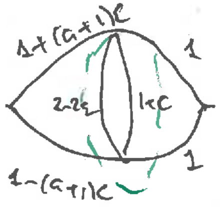
Consider



$$= \frac{C}{(p^2)^{1-\varepsilon}} \Rightarrow \frac{C}{p^2} =$$

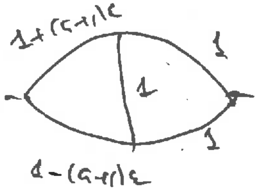


$=$



$$\frac{1}{\pi^{D/2} A(2-2\varepsilon, 1+(a+1)\varepsilon)} =$$

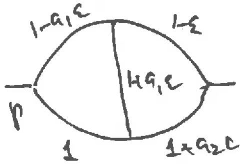
$$= \frac{A(2-2\varepsilon, 1+\varepsilon)}{A(2-2\varepsilon, 1+(a+1)\varepsilon)}$$



← Calculated by IBP

IV Additional loop

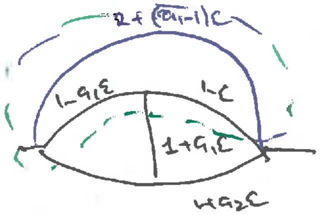
Consider



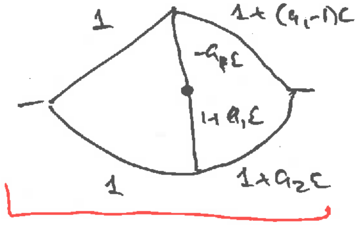
$$= \frac{C}{(p^2)^{1+(a_2+1)\varepsilon}} .$$

Integrate with additional propagator:

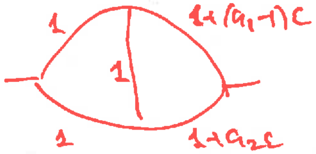
$$c\, A\left(1+(a_2+1)\varepsilon, 2+(a_1-1)\varepsilon\right) \frac{1}{p^2(1+(a_1+a_2-1)\varepsilon)} =$$



$$= A(1-a_1\varepsilon, 1-\varepsilon)$$



Calculated by IBP ↘



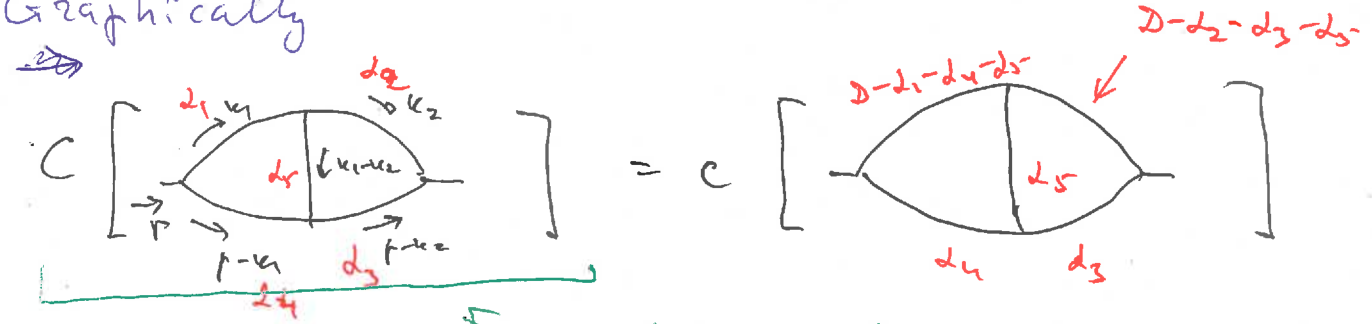
(V) Conformal transformation of inversion

All $k_i^\mu \to k_i^\mu \Rightarrow k_i^2 \Rightarrow \frac{1}{k_i^2}$ $(k_i-k_j)^2 \Rightarrow \frac{(k_i-k_j)^2}{k_i^2 k_j^2}$ (see also prove of uniquness relation), $d^Dk_i \to \frac{d^Dk_i}{(k_i^2)^D}$

$$\int \frac{d^Dk_1\, d^Dk_2}{k_1^{2\alpha_1} k_2^{2\alpha_2} (k_1-k_2)^{2\alpha_5} (p-k_1)^{2\alpha_4} (p-k_2)^{2\alpha_3}} \longrightarrow \int \frac{d^Dk_1\, d^Dk_2 \; k_1^{2\alpha_1} k_2^{2\alpha_2} (k_1^2k_2^2)^{2\alpha_5} (p^2k_1^2)^{2\alpha_4} (p^2k_2^2)^{2\alpha_3}}{k_1^{2D} k_2^{2D} (k_1-k_2)^{2\alpha_5} (p-k_1)^{2\alpha_4} (p-k_2)^{2\alpha_3}}$$

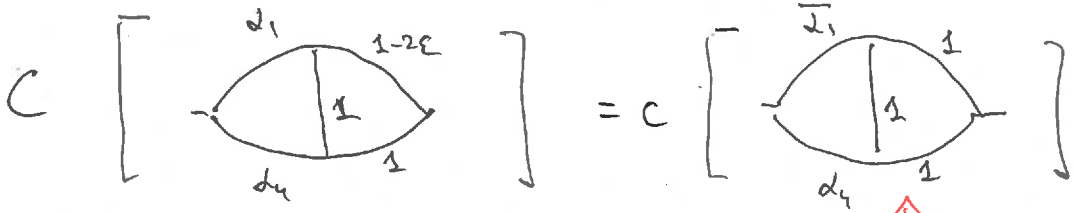
Graphically

$\Rightarrow$



↖ coefficient function : $I = C[I]\frac{1}{p^{2\alpha}}$

Example



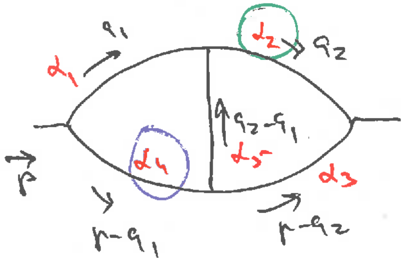
, where $\bar{\alpha}_1 = D-1-\alpha_1-\alpha_4$

↑ Calculated by IBP

VIII Dual diagrams and Fourier transform

In principle we can work in „x" space. To transform FI to „x" space we can use two possibilitiy

Ⓐ. Dual diagram : $p \to z$
$q_i \to x_i$



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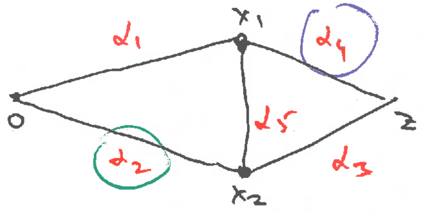
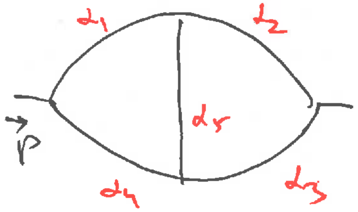


Integral is same but graphical representation is different.
For this case only the lines with α_2 and α_2 are exchanged.

Calculations in QCD [D.I. Kazakov, A.V.K., 1988, 1990]
Method of Gegenbauer Polynomials [A.V.K., 1996]
$N=4$ SYM [J.M. Drummond, J. Henn, G.P. Korchemsky, E. Sokatchev, 2010]

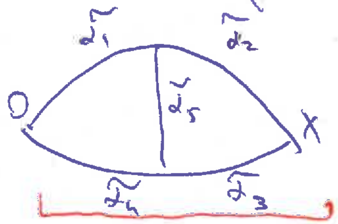
Ⓑ Fourier transform



$$= \frac{C}{(p^2)^{\alpha_F}} \qquad \alpha_F = \sum_1^5 \alpha_i - D \ , \quad \tilde{\alpha}_i = \frac{D}{2} - \alpha_i$$

If we do Fourier transfor of lhs and rhs parts we have

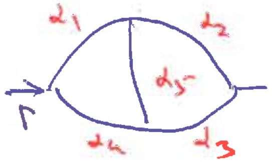
$$C = \frac{\prod_1^5 a(\alpha_i)}{a(\alpha_F)}$$



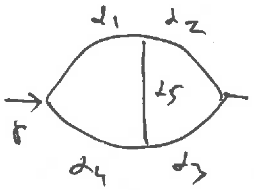
$$\tilde{C}\ \frac{1}{(x^2)^{\tilde{\alpha}_F}}$$

Because $a(\tilde{\alpha}) = 1/a(\alpha)$

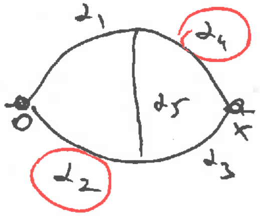
$$\tilde{C} = \frac{\prod_1^5 a(\tilde{\alpha}_i)}{a(\tilde{\alpha}_F)}$$



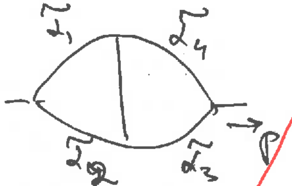
If we would like to continue to work in "p" space we can tranform the initial diagram to "x" space by dual [Fourier] transform and to return by Fourier [dual] transform :



$\overset{\Pi}{\rightleftarrows}$



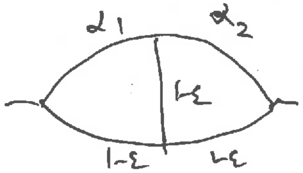
$$= \underbrace{\frac{\prod_1^5 a(\tilde{\alpha}_i)}{a(\tilde{\alpha}_p)}}$$



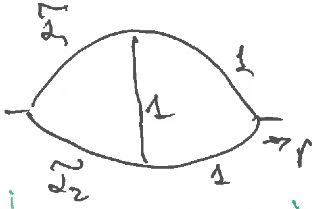
In general (the diagram has N lines and L loops):

$$\frac{\prod_1^N a(\alpha_i)}{a(\alpha_p)}, \quad \alpha_p = \sum_1^N \alpha_i - \frac{L}{2} D$$

Example



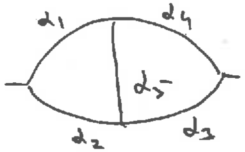
$$= \frac{\prod_1^2 a(\alpha_i) \cdot a^3(1-\varepsilon)}{a(\alpha_1+\alpha_2-1\bullet\varepsilon)}$$



$$\tilde{\alpha}_i = D/2 - \alpha_i$$

calculated by IBP

IV General result



$$\overset{\alpha_i = 1 + a_i\varepsilon}{=\!=} \frac{K_2}{1-2\varepsilon}\left\{A_0\zeta_3 + A_1\zeta_4\varepsilon + A_2\zeta_5\varepsilon^2 + [A_3\zeta_6 - A_4\zeta_3^2]\varepsilon^3 + O(\varepsilon^4)\right\}$$

[I. Bierenbaum, S. Weinzierl, 2003]

[Kazakov, 1985]

$$K_n = \exp\left[-n\left(\gamma_E\varepsilon + \frac{\zeta(2)}{2}\varepsilon^2\right)\right], \quad A_0 = 6,\ A_1 = 9,\ A_3 = \frac{5}{2}(A_2 - 6), \quad A_n = a_1^n + a_2^n + a_3^n + a_4^n$$

$$A_2 = 42 + 30A + 45a_5 + 10A_2 + 15a_5^2 + 15a_5A + 10(a_1a_2 + a_3a_4 + a_1a_4 + a_2a_5) + 5(a_1a_3 + a_2a_4)$$

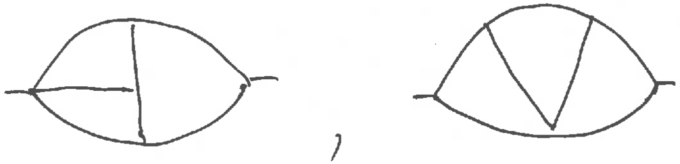
$$A_4 = 46 + 42A + 45a_5 + 14A_2 + 15a_5^2 + 33a_5A + 50(a_1a_2 + a_3a_4) + 31(a_1a_3 + a_2a_4) + 14(a_1a_4 + a_2a_3)$$

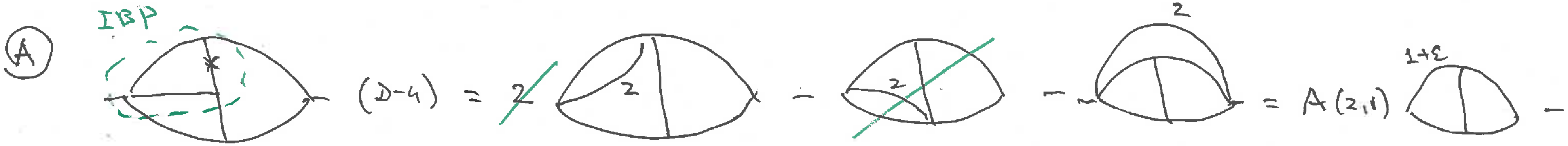
$$+ 6a_5A_2 + 6a_5^2A + 24a_5(a_1a_2 + a_3a_4) + 12a_5(a_1a_3 + a_2a_4) + 12(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4)$$

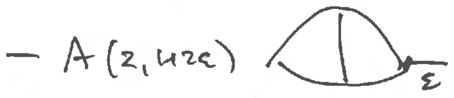
$$+ 12(a_1^2a_2 + a_2^2a_1 + a_3^2a_1 + a_4^2a_3) + 6(a_1^2a_3 + a_3^2a_1 + a_3^2a_4 + a_4^2a_2)$$

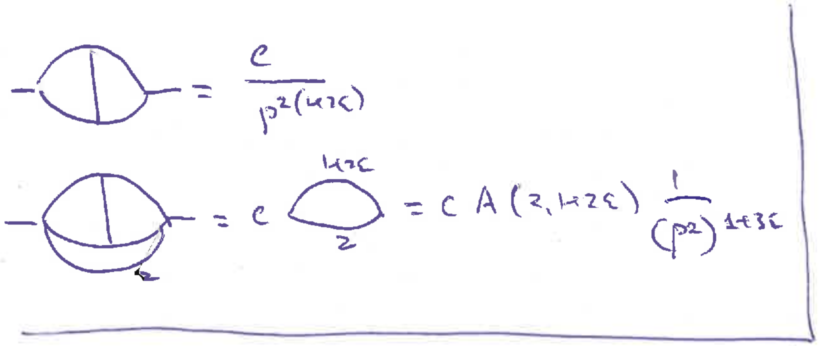
X The-loop case

Two basic topology



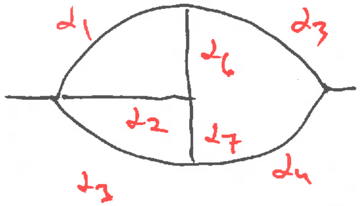






So we transform 3-loop diagrams to two loop ones.

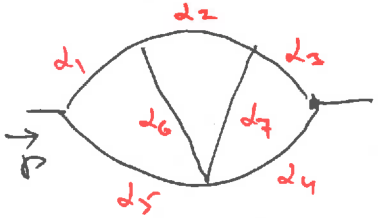
Kazakov :



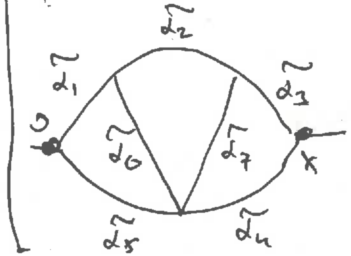
$$\overset{\alpha_i=1+a_i\varepsilon}{=} \frac{K_3}{1-2\varepsilon}\left[20\zeta_5+\varepsilon\left\{50\zeta_6+\left(20+6\sum_4^7 a_i\right)\zeta_3^2\right\}+O(\varepsilon^2)\right]$$

[D.I. Kazakov, 1985]

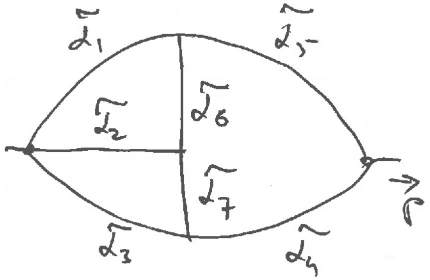
Consider



$$= F \, \frac{\prod_{k=1}^{7} a(\alpha_i)}{a\left(\sum_i^7 \alpha_i - \frac{3}{2} D\right)} \Bigg[$$



$$\overset{\sqcap}{=}$$



$$\Bigg] =$$

it is considered already

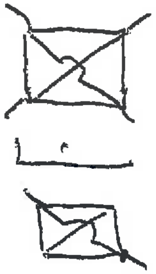
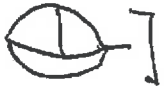
$$\underset{\tilde{\alpha}_i = 1-(a_i+1)\varepsilon}{\overset{\alpha_i = 1 + a_i \varepsilon}{=}} \frac{K_3}{(1-2\varepsilon)} \underbrace{\left[20 \zeta_5 + \varepsilon \left\{ 50 \zeta_6 - \left(4 + 6 \sum_i^7 a_i\right) \zeta^2(\varepsilon) \right\} + O(\varepsilon^2) \right]}_{\text{obtained from the previous one : } a_i \to -(1+a_i)}$$

XI

~~Lectures~~ ㉒ ~~21~~

To calculate anomalous dimensions and β-functions really, it is necessary to evaluate (often) only the singular structure of diagrams. It is often gives a possibility to obtain results at high orders. The important property: the absence of p-dependence in KR' of the diagrams (here = in singularities of the diagrams)

Examples

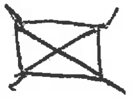
① Sing [] = Sing [] = ⊛

$= C A(1,4+2\varepsilon)\, \underline{3\varepsilon}$

$$A(1,4+2\varepsilon) = \frac{\Gamma(1-\varepsilon)\Gamma(1+3\varepsilon)\,\Gamma(3\varepsilon)}{\Gamma(1)\,\Gamma(4+2\varepsilon)\,\Gamma(2+4\varepsilon)} = \frac{1}{3\varepsilon} + \text{finite}$$

⊛ $= \frac{1}{3\varepsilon}$ finite  $= \frac{2}{\varepsilon}\zeta(3)$

6ζ(3)

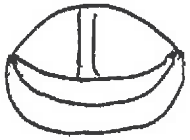
② Sing [] = Sing [] $= ⊛ \frac{1}{4\varepsilon} \underbrace{\text{finite}}_{20\,\zeta(5)}$  $= \frac{5}{\varepsilon}\zeta(5)$

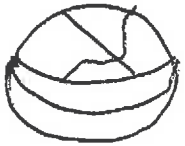
||

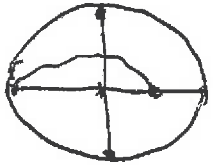
 = c $\underset{1+3\varepsilon}{\longleftarrow}$ ⇒  = c  $= c\,\underbrace{A(1,1+3\varepsilon)}_{\approx \frac{c}{4\varepsilon}+\text{finite}}\ \underline{4\varepsilon}$

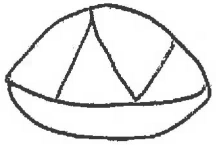
β-function (5-loop) in φ^4-model in D=4 Chetyrkin et al., Phys. Lett. B 132 (1983) 351

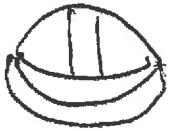
4-diagrams have been calculated numerically


(a)

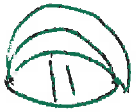

(b)


(c)


(d)

Ⓐ Sing [] = Ⓐ

let  = C $\overline{}$ $8-\frac{3}{2}D = 2+3\varepsilon$

 = C 

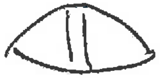
= C A(1,1) 

$= c\,\underbrace{A(1,1)\,A(\varepsilon,2+3\varepsilon)}\leftarrow 5\varepsilon$

$$\hookrightarrow = \frac{\Gamma^2(1-\varepsilon)\cancel{\Gamma(\varepsilon)}}{\Gamma^2(1)\cancel{\Gamma(2-2\varepsilon)}}\;\frac{\cancel{\Gamma(2-2\varepsilon)}\,\Gamma(-4\varepsilon)\,\Gamma(5\varepsilon)}{\cancel{\Gamma(\varepsilon)}\,\Gamma(2+3\varepsilon)\,\Gamma(2-6\varepsilon)} = -\frac{1}{20\varepsilon^2}\,\frac{1}{(1+3\varepsilon)(1-6\varepsilon)}\,\underbrace{\frac{\Gamma^2(1-\varepsilon)\Gamma(1-4\varepsilon)\Gamma(1+5\varepsilon)}{\Gamma(1+3\varepsilon)\Gamma(1-6\varepsilon)}}_{\Gamma^2(1-\varepsilon)\,(1+O(\varepsilon^2))}$$

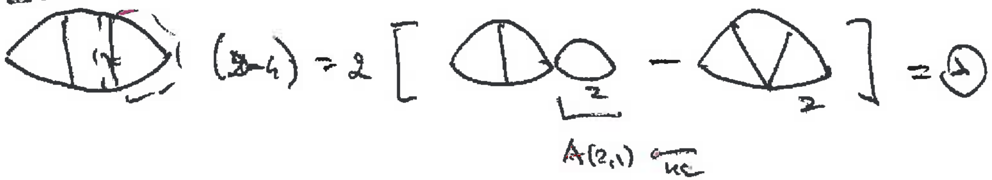
$\rightarrow \mu^2 \rightarrow \bar{\mu}^2$

Ⓐ = Sing $\Big[-\frac{1}{20\varepsilon^2}\,\frac{1}{(1+3\varepsilon)(1-6\varepsilon)}\cdot\Big\{$  with the accuracy $O(\varepsilon)\Big\}\Big]$

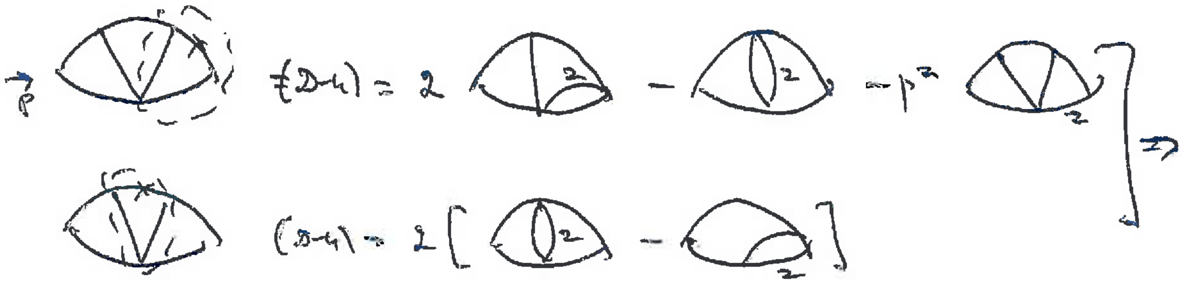
Consider 

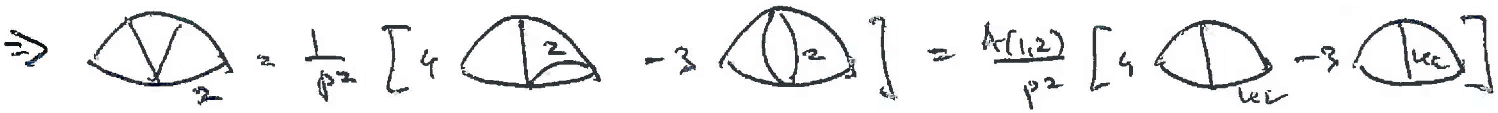
IBP

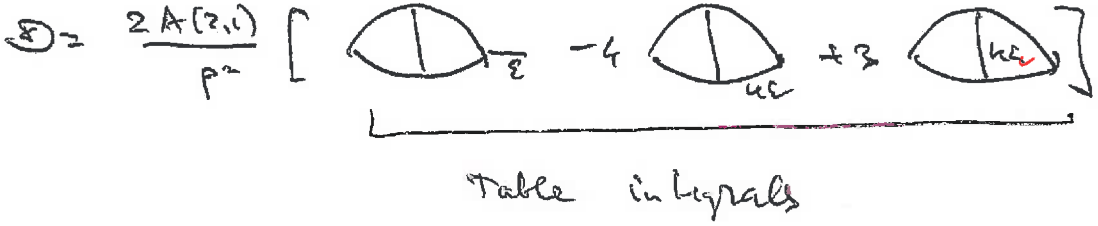
IBP



Consider

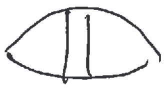






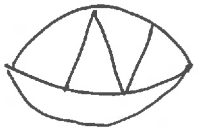
In principle,
where is full ε-dependence
(exact result) because

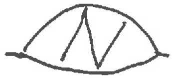
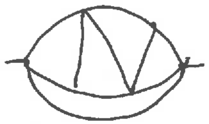
$\sim \zeta_2 (\dots 1)$

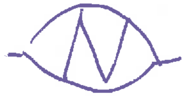


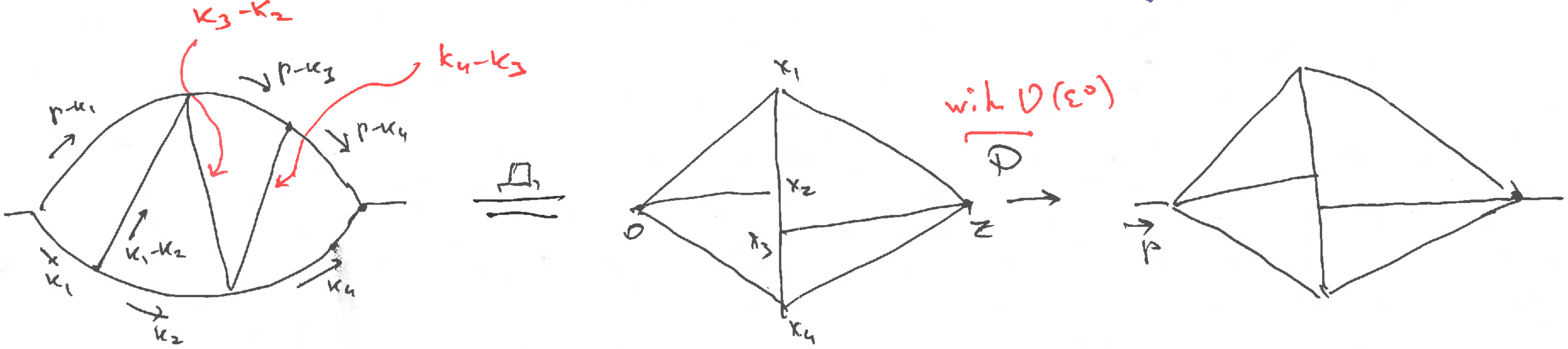
$$\Big|_{p^2=1} = \frac{K_3}{1-2\varepsilon}\left[20\zeta_5 + [50\zeta_6 + 44\zeta_3^2]\varepsilon + [317\zeta_7 + 132\zeta_2\zeta_3]\varepsilon^2\right]$$

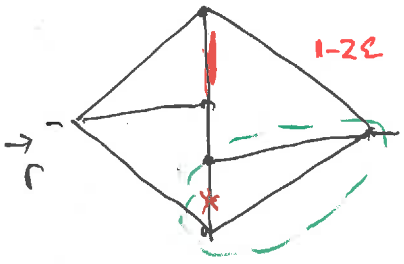
$$K_n = \exp\left[-n\left(\gamma_E\varepsilon + \frac{\zeta_2}{2}\varepsilon^2\right)\right]$$

Sing []

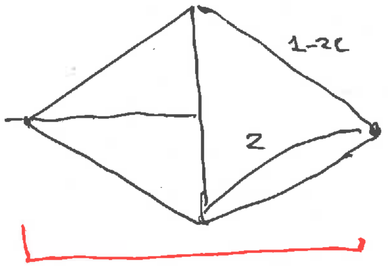
Let  $= \frac{A}{p^{2\alpha}}$, $\alpha = 9 - 4\cdot\frac{D}{2} = 9 - 2D = 1+4\varepsilon$ $\Rightarrow$  $= A \cdot A(1,1+4\varepsilon) = \frac{A}{5\varepsilon}$,

So we should found  with $O(\varepsilon^0)$ accuracy





$(D-4) = 2$



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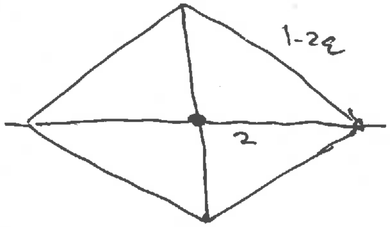

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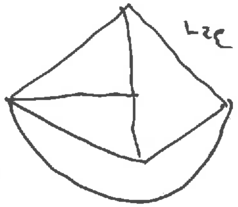
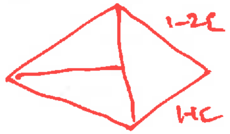
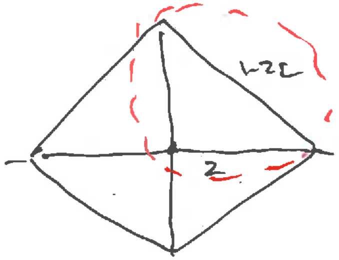


$= ⊛$

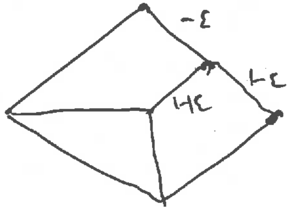
$A(2,1)$



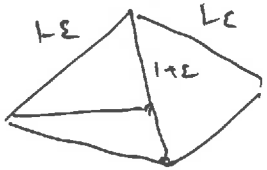
Uni: Δ

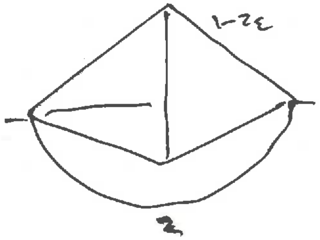


$= A(2,1)$

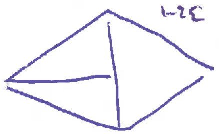


$= A(2,1)$



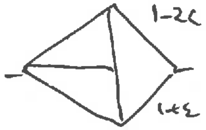


$= A \cdot A(1+\varepsilon,2)$



$= \frac{A}{p^{2\alpha}}$, $\alpha = 7-2\varepsilon-3(2-\varepsilon) = 1+\varepsilon$

$⊛ = A(2,1)\Big[\, 2$



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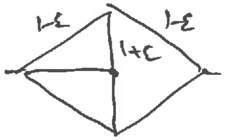
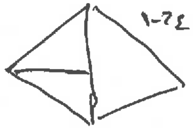


$-$

$- \frac{A(2,1+\varepsilon)}{A(2,1)}$



$\Big] = ⊛$

$$A(2,1) = \frac{\Gamma(-\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2)\Gamma(1)\Gamma(1-2\varepsilon)} = -\frac{1}{\varepsilon}\,\frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}$$

$$A(2,1+\varepsilon) = \frac{\Gamma(-\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma(2)\Gamma(1+\varepsilon)\Gamma(1-3\varepsilon)} = -\frac{1}{\varepsilon}\,\frac{\Gamma(1-\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(1-3\varepsilon)}$$

$\Rightarrow$

$$\Rightarrow \frac{A(2,1+\varepsilon)}{A(2,1)} = \frac{\Gamma^2(1-2\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1-\varepsilon)\Gamma(1-3\varepsilon)} = \frac{\exp\left[\cancel{\gamma_E\varepsilon\left(2\cdot 2-2\right)} + \sum_{n=2}^{\infty}\frac{\zeta(n)}{n}\varepsilon^n\left[2\cdot 2^n+(-2)^n\right]\right]}{\exp\left[\cancel{\gamma_E\varepsilon\left(2\cdot(-1)+3+1\right)} + \sum_{n=2}^{\infty}\frac{\zeta(n)}{n}\varepsilon^n\left[1+3^n+(-2)^n\right]\right]} =$$

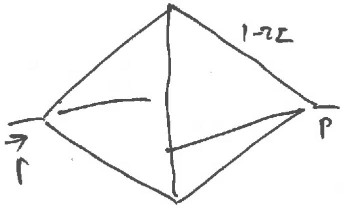
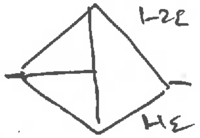
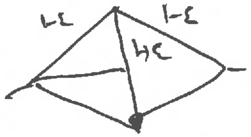
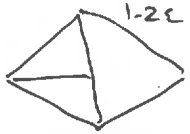
$$\Gamma(1+x) = \exp\left[-\gamma_E x + \sum_{n=2}^{\infty}\frac{\zeta(n)}{n}(-x)^n\right]$$

$$= \exp\left[\sum_{n=2}^{\infty}\frac{\zeta(n)}{n}\varepsilon^n p(n)\right] \qquad p(n) = 2^{n+1}-1-3^n+2(-)^n\left[2^{n-1}-1\right]$$

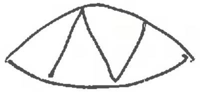
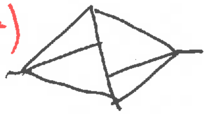
$$\simeq \cancel{1+}\, O(\varepsilon^3)$$

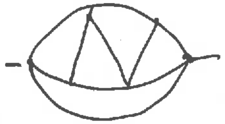
$p(2) = 8-1-9+2 = 0$

$p(3) = \underline{16}-1-27-\underline{6} = 10-28 = -18$

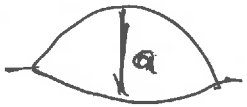
 $\cdot (D-4) = A(2,1) \Big[2$  $-$  $-$  $\Big]$

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{Table integrals}}$

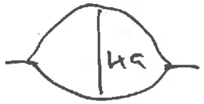
After evaluation :  $\overset{O(\varepsilon^1)}{=}$  $= \frac{441}{8}\zeta_7 + O(\varepsilon^1)$

So $\quad \mathrm{Sing}\Big[$  $\Big] = \frac{441}{40}\frac{\zeta_7}{\varepsilon} +$ (see also, P.A.Baikov, K.G. Chetyrkin, 2010)

Really to ~~have~~ ~~obtain~~ obtain the result we should have longer expansion of the table integrals.

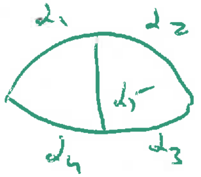
The additional coefficients of expansion can be obtained from better knowledge of the diagram 

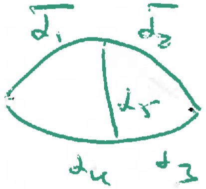
XII Functional Equations [Kazakov, 1985]

Consider 

$\equiv F(1+a)$

Remember:

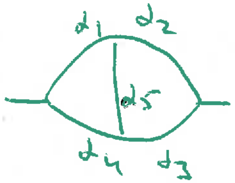
① C [

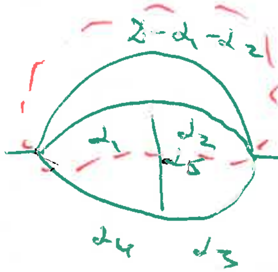
] $\xlongequal{\text{inversion}}$ C [

]

$\bar{\alpha}_1 = D-\alpha_1-\alpha_4-\alpha_5$

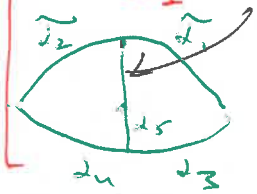
$\bar{\alpha}_2 = D-\alpha_2-\alpha_3-\alpha_5$

② additional loop 

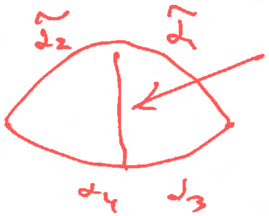
$= \dfrac{1}{\underbrace{A\left(D-\alpha_1-\alpha_2,\ \sum_1^5 \alpha_i - D\right)}_{A_1}}$ 

$=$

Uni Δ

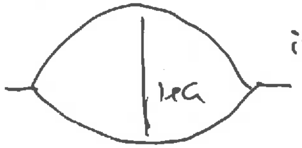
$= \dfrac{A(\alpha_1,\alpha_2)}{A_1}$ [

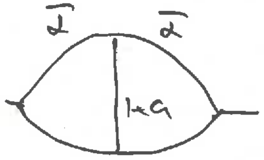
$\alpha_1+\alpha_2-\frac{D}{2}$

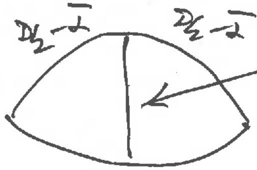
$=$ 

$\alpha_1+\alpha_2+\alpha_5-\frac{D}{2}$

$\tilde{\alpha}_i = \frac{D}{2}-\alpha$]



$\xlongequal{\text{inver}}$ 

$\xlongequal{\text{add. loop}}$ 

$\bar{\alpha}_5 = 2(1-a-2a)+1+a-2+\varepsilon = 1-a-3\varepsilon$

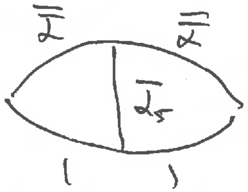
$=$

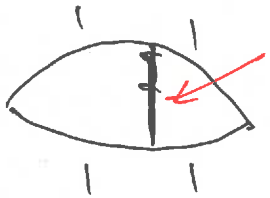
$\bar{\alpha} = D-3-a = 1-a-2\varepsilon$

$\frac{D}{2}-\bar{\alpha} = 1+a+\varepsilon$

$\bar{\alpha} =$ ~~$\frac{D}{2}(\frac{D}{2}-\bar{\alpha})-$~~

$D-(1+a+\varepsilon)-\bar{\alpha}_5 = D-1-(2-2\varepsilon) = 1$

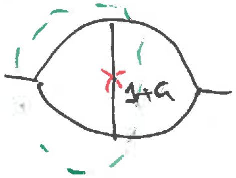
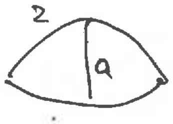
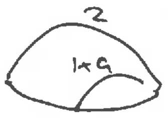
$\xlongequal{\text{inver}}$ 

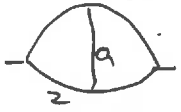
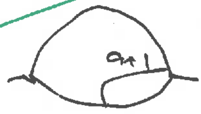
$=$ 

$1-a-3\varepsilon$

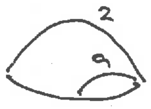
$$\boxed{F(1+a) = F(1-a-3\varepsilon) \qquad (1)}$$

IBP

 $\underbrace{(D-4-2a)}_{-2(a+\varepsilon)} = 2\Big[$  $-$  $\Big]$ (2.1)

 $\underbrace{(D-3-a)}_{1-2\varepsilon-a} =$  $-p^2$  $+a\Big[$  $-$  $\Big]$ (2.2)

$\varepsilon\cdot(2.1) + (2.2)$

$-(a+\varepsilon)F(1+a) + (1-2\varepsilon-a)F(a) =$  $-$  $= \underbrace{A(a,2)A(1,1+a+\varepsilon) - A(1,a+1)A(2,1+a+\varepsilon)}_{\text{F-functions}}$

After little algebra

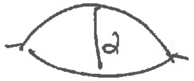
$$\boxed{F(1+a) = \frac{1-2\varepsilon-a}{a+\varepsilon}F(a) + \frac{2(2a-1+3\varepsilon)\Gamma(-a-\varepsilon)\Gamma(a+2\varepsilon)\Gamma^2(1-\varepsilon)}{(a+\varepsilon)\Gamma(a+1)\Gamma(1-3\varepsilon-a)}} \qquad (2)$$

Solution of the system of equations (1) and (2):

$$F(1+a) = 2\,\frac{\Gamma^2(1-a)\,\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}\left[\frac{\Gamma(1-\varepsilon)\,\Gamma(1+\varepsilon)\,\Gamma(-a-2\varepsilon)\,\Gamma(a+2\varepsilon)}{\Gamma(1-2\varepsilon)\,\Gamma(1+2\varepsilon)} + \frac{\Gamma(-a-\varepsilon)\,\Gamma(a+2\varepsilon)}{\Gamma(1+a)\,\Gamma(1-a-3\varepsilon)}\,\times\right.$$

$$\left.\times\,\underbrace{\sum_{n=0}^{\infty}(-1)^n\,\frac{\Gamma(n+1-2\varepsilon)}{\Gamma(n+\varepsilon)}\left(\frac{1}{n+a+\varepsilon}+\frac{1}{n-a-2\varepsilon}\right)}_{\text{Two hypergeometric functions } {}_3F_2(\dots|-1)}\right]$$

with the argument -1.

$\overline{\text{XIII}}$ Another approach, useful for  is ~~B~~based on Gegenbaer Polynomials (GP) ㉜
[K.G. Chetyrkin, A.L. Kataev, F.V. Tkachov, 1980]
[A.V.K. 1996]

|| GP gives a possibility to cancel one propagator of a diagram
Here we will follow to

① $$\left(1-2\underbrace{rt}_{\cos\theta}+r^2\right)^{-\delta}=\sum_{n=0}^{\infty}\underset{\uparrow_{GP}}{C_n^{\delta}(t)}\,r^n \quad (r\leq 1)$$ ← definition

② Application for propagators

$$\frac{1}{[(p_1-p_2)^2]^{\delta}}=\frac{1}{[p_1^2-2|p_1||p_2|\cos(\widehat{p_1p_2})+p_2^2]^{\delta}}=\left[\frac{\Theta(p_1^2>p_2^2)}{(p_1^2)^{\delta}}\sum_{n=0}^{\infty}C_n^{\delta}\Big(\underbrace{\cos(\widehat{p_1p_2})}_{\hat{p}_1\hat{p}_2}\Big)\left(\frac{p_2^2}{p_1^2}\right)^{n/2}\right]+\left[1\leftrightarrow 2\right]$$

③ $\lambda=D/2-1$ ← special choise

$$\int d\hat{p}_2\, C_n^{\lambda}(\hat{p}_1\hat{p}_2)\, C_m^{\lambda}(\hat{p}_2\hat{p}_3)=\frac{\lambda}{n+\lambda}\,\delta_n^m\, C_n^{\lambda}(\hat{p}_1\hat{p}_3)$$ ← orthogonality in D-space

Remember: $d^Dx=\frac{1}{2}S_{D-1}(x^2)^{\lambda}dx^2d\hat{x}$, $S_{D-1}=\frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)}$ - surface of the unit hypersphere in R_D

④

$$\left.\begin{array}{l} C_n^{\delta}(x) = \displaystyle\sum_{p=0}^{[n/2]} \frac{(-1)^p\,\Gamma(n-p+\delta)\,(2x)^{n-2p}}{(n-2p)!\,p!\,\Gamma(\delta)} \\[2em] \dfrac{(2x)^n}{n!} = \displaystyle\sum_{p=0}^{[n/2]} C_{n-2p}^{\delta}(x)\,\frac{(n-2p+\delta)\,\Gamma(\delta)}{p!\,\Gamma(n-p+\delta+1)} \end{array}\right] \Longrightarrow$$

$$C_n^{\delta}(x) = \sum_{k=0}^{[n/2]} C_{n-2k}^{\lambda}(x)\,\frac{(n-2k+\lambda)\,\Gamma(\lambda)}{k!\,\Gamma(\delta)}\,\frac{\Gamma(n+\delta-k)\,\Gamma(k+\delta-\lambda)}{\Gamma(n+\lambda+1-k)\,\Gamma(\delta-\lambda)}$$

If we expand the propagator with the index "λ" we will have one-fold series. For the general index "δ" we will have two-fold series.

⑤ It is convenient to work with traceless products: $x^{\mu_1\ldots\mu_n}$

$$\left(x^{\mu_1\mu_2} = x^{\mu_1}x^{\mu_2} - \frac{g_{\mu_1\mu_2}}{D}x^2 \Rightarrow g_{\mu_1\mu_2}x^{\mu_1\mu_2} = 0\right)$$

$$C_n^\lambda(\hat{x}\hat{z}) = \frac{2^n\Gamma(n+\lambda)}{n!\,\Gamma(\lambda)}\,\frac{x^{\mu_1\ldots\mu_n}z^{\mu_1\ldots\mu_n}}{[x^2z^2]^{n/2}}$$

The traceless product does not complicate calculations:

$$\int d^Dx\,\frac{x^{\mu_1\ldots\mu_n}}{x^{2\alpha}(x-y)^{2\beta}} = \pi^{D/2}A^{n,0}(\alpha,\beta)\,\frac{y^{\mu_1\ldots\mu_n}}{y^{2(\alpha+\beta-D/2)}} \quad \text{(d1)}\quad,\quad A^{n,m}(\alpha,\beta) = \frac{a_n(\alpha)\,a_m(\beta)}{a_{n+m}(\alpha+\beta-D/2)}\,,\quad a_n(\alpha) = \frac{\Gamma(\tilde{\alpha}+n)}{\Gamma(\alpha)}$$

!!! But we have Θ- function !!!

⑥

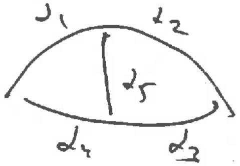
$$\int d^Dx \, \frac{x^{\mu_1\ldots\mu_n}}{x^{2\alpha}(x-y)^{2\beta}} \, \theta(x^2-y^2) = \frac{\pi^{D/2} y^{\mu_1\ldots\mu_n}}{y^{2(\alpha+\beta-\frac{D}{2})}} \sum_{m=0}^{\infty} \frac{B(m,n|\beta,\lambda)}{m+\alpha+\beta-\frac{D}{2}} \overset{\beta=\lambda}{=\!=} \frac{\pi^{\frac{D}{2}} y^{\mu_1\ldots\mu_n}}{y^{2\underbrace{(\alpha+\beta-\frac{D}{2})}_{\alpha-1}}} \frac{1}{\Gamma(\lambda)} \frac{1}{(\alpha-1)(n+\lambda)} \qquad (d2)$$

$$\int d^Dx \, \frac{x^{\mu_1\ldots\mu_n}}{x^{2\alpha}(x-y)^{2\beta}} \, \theta(y^2-x^2) = \frac{\pi^{\frac{D}{2}} y^{\mu_1\ldots\mu_n}}{y^{2(\alpha+\beta-\frac{D}{2})}} \sum_{m=0}^{\infty} \frac{B(m,n|\beta,\lambda)}{m+n-\alpha+\frac{D}{2}} \overset{\beta=\lambda}{=\!=} \frac{\pi^{\frac{D}{2}} y^{\mu_1\ldots\mu_n}}{y^{2(\alpha-1)}} \frac{1}{\Gamma(\lambda)} \frac{1}{(n+\frac{D}{2}-\alpha)(n+\lambda)}, \qquad (d3)$$

where

$$B(m,n|\beta,\lambda) = \frac{\Gamma(m+\beta+n)}{m!\,\Gamma(m+n+1+\lambda)\,\Gamma(\beta)} \, \frac{\Gamma(m+\beta-\lambda)}{\Gamma(\beta-\lambda)}$$

The sum of (d2) and (d3) $\longrightarrow$ (d1) : coming from relations between hypergeometric functions

Using GP method, the results for the diagram  with 3 arbitrary indices have been obtained. They are complicated; they contain 4 ${}_3F_2(\ldots|1)$.

But the result for $I(\alpha)$ is single

$I_\alpha =$  $= -2\,\frac{\Gamma^2(\lambda)\Gamma(\lambda-\alpha)\Gamma(\alpha+1-2\lambda)}{\Gamma(2\lambda)\Gamma(3\lambda-\alpha-1)} \cdot \Big[\frac{\Gamma^2(\frac{\lambda}{2})\Gamma(3\lambda-\alpha-1)\Gamma(2\lambda-\alpha)\Gamma(\alpha+1-2\lambda)}{\Gamma(\lambda)\Gamma(2\lambda+\frac{\lambda}{2}-a)\Gamma(\frac{\lambda}{2}-2\lambda+\alpha)}$

$+ \underbrace{\sum_{n=0}^{\infty} \frac{\Gamma(n+2\lambda)}{\Gamma(m+\alpha+1)} \frac{1}{n+1+\lambda+\alpha}}_{{}_3F_2(\ldots\,(1)} \Big]$

Comparing the results with Kazakov one, we see the relation between one ${}_3F_2(\ldots|1)$ and two ${}_3F_2(\ldots|-1)$. Now the relation is proven. [A.V.K. and S.Teber, 2016]

Teber

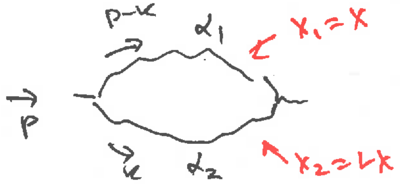
Now I will come to consideration of diagrams with massive propagators.

I would like to note the recent paper [D.A. Baikov, K.G. Chetyrkin] where successful expansion of master integrals has been done. [3-loop and 4-loop]

Calculation FD with masses (into propagators)

We can repeat the rules and definition on pages (1-5):

I. Consider one-loop



$$= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 dx \cdot x^{\alpha_1-1}(1-x)^{\alpha_2-1} \int \frac{d^D k}{\underbrace{[\ ]_1^{\alpha_1+\alpha_2}}} = \circledast$$

$$[\ ]_1 = x\left[(k-p)^2+m_1^2\right] + (1-x)\left[k^2+m_2^2\right] = k^2+p^2 x - 2(pk)x + m_1^2 x + m_2^2(1-x)$$

$$\int \frac{d^D k}{[\ ]_1^{\alpha_1+\alpha_2}} = \pi^{D/2} \frac{\Gamma(\alpha_1+\alpha_2-D/2)}{\Gamma(\alpha_1+\alpha_2)} \frac{1}{\left[m_1^2 x + m_2^2(1-x) + p^2 x(1-x)\right]^{\alpha_1+\alpha_2-D/2}}$$

$$\circledast = \pi^{D/2} \frac{\Gamma(\alpha_1+\alpha_2-D/2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 \frac{dx \cdot x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{\left[m_1^2 x + m_2^2(1-x) + p^2 x(1-x)\right]_2^{\alpha_1+\alpha_2-D/2}} = \circledast$$

Particular cases

Ia) $\underline{m_1=m_2=0}$ Consider on page 6

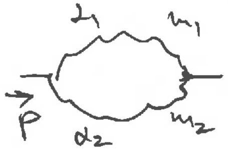
Ib) $\underline{m_2^2=0\,,\ m_1^2=-p^2\equiv m^2}$ $\boxed{\text{On-shell}}$

$$[\;]_2 = m_1^2 x + p^2 x(1-x) = m^2 x^2 \qquad \int_0^1 \frac{dx\cdot x^{\beta-1}(1-x)^{\alpha_2-1}}{\{[\;]_2 = m^2x^2\}^{\alpha_1+\alpha_2-\frac{D}{2}}} = \frac{1}{(m^2)^{\alpha_1+\alpha_2-\frac{D}{2}}}\underbrace{\int_0^1 dx\cdot x^{\beta-1}(1-x)^{\alpha_2-1}}_{\frac{\Gamma(\beta)\Gamma(\alpha_2)}{\Gamma(\beta+\alpha_2)}}$$

$$\beta = \alpha_1-2\left(\alpha_1+\alpha_2-\frac{D}{2}\right) = D-\alpha_1-2\alpha_2\ ,\quad \beta+\alpha_2 = D-\alpha_1-\alpha_2$$

$$\circledast = \pi^{D/2}\underbrace{\frac{\Gamma(\alpha_1+\alpha_2-\frac{D}{2})}{\Gamma(\alpha_1)\cancel{\Gamma(\alpha_2)}}\;\frac{\cancel{\Gamma(\alpha_2)}\,\Gamma(D-\alpha_1-2\alpha_2)}{\Gamma(D-\alpha_1-\alpha_2)}}_{C(\alpha_1,\alpha_2)}\qquad\frac{1}{(m^2)^{\alpha_1+\alpha_2-\frac{D}{2}}}$$

II.



$$= \pi^{d/2} \frac{\Gamma(\alpha_1+\alpha_2-\frac{d}{2})}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 \frac{dx \cdot x^{\tilde{\alpha}_1-1} (1-x)^{\tilde{\alpha}_2-1}}{\Big[p^2 + \underbrace{\frac{m_1^2}{1-x} + \frac{m_2^2}{x}}_{\mu^2}\Big]^{\alpha_1+\alpha_2-\frac{d}{2}}}$$

$$\int_0^1 dx \cdot x^{\tilde{\alpha}_1-1}(1-x)^{\tilde{\alpha}_2-1}$$

One loop can be represented as the integral from a new propagator with effective "mass" dependent on X

If $m_1^2=0 \Rightarrow \mu^2 = \frac{m_2^2}{x}$

If $m_1^2=m_2^2=m^2 \Rightarrow \mu^2 = \frac{m^2}{x(1-x)}$

It is convenient to use the inverse mass expansion (no Mellin-Barnes integrals !!!)

[So, we can decrease number of logs in some FI. !!!]

Example Two loop tadpole with one massless line and two massive lines with $m_1 = m_2 = m$

$$= \frac{\Gamma(\alpha_1+\alpha_2-\frac{D}{2})}{\Gamma(\alpha_2)\Gamma(\alpha_3)} \cdot \pi^{\frac{D}{2}} \cdot \int_0^1 dx \cdot x^{\tilde{\alpha}_2-1}(1-x)^{\tilde{\alpha}_3-1}$$

$\mu^2 = \frac{m^2}{x(1-x)}$

$$\underbrace{\qquad\qquad}_{\pi^{\frac{D}{2}} B(\alpha_1,\beta) \frac{1}{(\mu^2)^{\beta+\alpha_1-\frac{D}{2}}}}$$

$=$ the integral on x is trivial

$\beta = \alpha_1+\alpha_2-\frac{D}{2}$

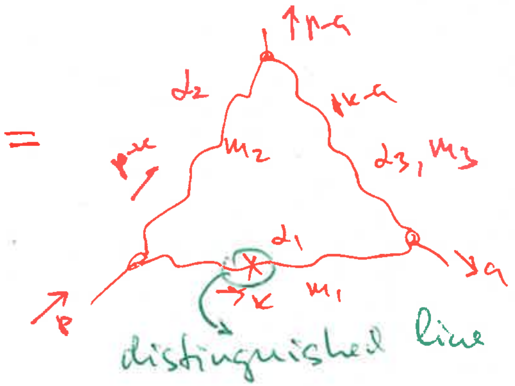
$$= \pi^D \underbrace{\frac{\Gamma(\frac{D}{2}-\alpha_1)\Gamma(\alpha_1+\alpha_3-\frac{D}{2})\Gamma(\alpha_1+\alpha_2-\frac{D}{2})\Gamma(\alpha_2+\alpha_3-D)}{\Gamma(\frac{D}{2})\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(2\alpha_1+\alpha_2+\alpha_3-D)}}_{F(\alpha_1,\alpha_2,\alpha_3)}$$

III IBP procedure

$$\int \frac{d^D k \quad \left[D = \frac{d}{dk_\mu} k_\mu \right]}{\left\{[k^2+m_1^2]^{\alpha_1}\,[(k-p)^2+m_2^2]^{\alpha_2}\,[(k-q)^2+m_3^2]^{\alpha_2}\right\}} = \int d^D k \left[\underbrace{\frac{d}{dk_\mu}\left(\frac{k^\mu}{\{\,\}}\right)}_{\substack{=0 \\ \text{Stoks theorem}}} - k^\mu \frac{d}{dk^\mu}\frac{1}{\{\,\}} \right]$$

$\equiv$

I =



$$-k_\mu \frac{d}{dk_\mu} \frac{1}{\underbrace{[k^2+m_1^2]^{\alpha_1}}_{[\,]_1^{\alpha_1}}} = +2\alpha_1 \frac{k^2}{[\,]_1^{\alpha_1+1}} = 2\alpha_1 \left[\frac{1}{[\,]_1^{\alpha_1}} - \frac{m_1^2}{[\,]_1^{\alpha_1+1}} \right]$$

$$-k_\mu \frac{\partial}{\partial k_\mu} \frac{1}{\underbrace{[(k-p)^2+m_2^2]^{\alpha_2}}_{[\,]_2^{\alpha_2}}} = \overbrace{2\alpha_2 \frac{(k,k-p)}{[\,]_2^{\alpha_2+1}}}^{\alpha_2[k^2+(k-p)^2-p^2]} = \alpha_2 \left[\frac{k^2+m_1^2}{[\,]_2^{\alpha_2+1}} + \frac{1}{[\,]_2^{\alpha_2}} - \frac{(p^2+m_1^2+m_2^2)}{[\,]_2^{\alpha_2+1}} \right]$$

Similar $$-k_\mu \frac{d}{dk_\mu} \frac{1}{\underbrace{[(k-q)^2+m_3^2]^{\alpha_3}}_{[\,]_3^{\alpha_3}}} = \alpha_3 \left[\frac{k^2+m_1^2}{[\,]_3^{\alpha_3+1}} + \frac{1}{[\,]_3^{\alpha_3}} - \frac{(p^2+m_1^2+m_3^2)}{[\,]_2^{\alpha_3+1}} \right]$$

So we have

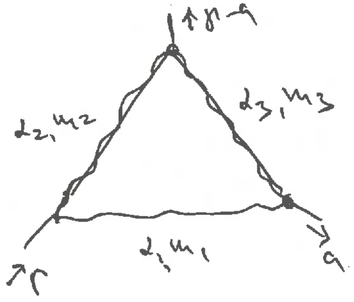
$$D I + \Big[-2\alpha_1 \{ I - m_1^2 I(\alpha_1\to\alpha_1+1) \} - 2\alpha_2 \Big\{ I\begin{pmatrix}\alpha_1\to\alpha_1-1\\ \alpha_2\to\alpha_2+1\end{pmatrix} + I - (p^2+m_1^2+m_2^2)\, I(\alpha_2\to\alpha_2+1) \Big\}_2 - 2\alpha_3 \{\alpha_2\leftrightarrow\alpha_3\}_2 \Big] = 0$$

or

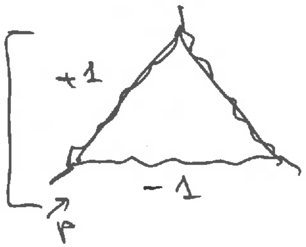
$$(D-2\alpha_1-\alpha_2-\alpha_3)\, I = \alpha_2 \Big\{ I\begin{pmatrix}\alpha_1\to\alpha_1-1\\ \alpha_2\to\alpha_2+1\end{pmatrix} - (p^2+m_1^2+m_2^2)\, I(\alpha_2\to\alpha_2+1) \Big\}_a + \alpha_3 \{\alpha_3\leftrightarrow\alpha_2\}_a$$

$$-2\alpha_1 m_1^2\, I(\alpha_1\to\alpha_1+1)$$

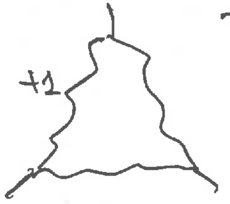
Graphically



$(D-2\alpha_1-\alpha_2-\alpha_3) = \alpha_2 \Big[$

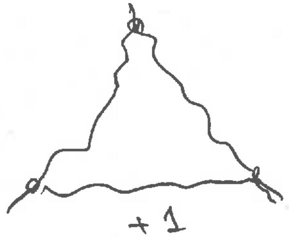


$-(p^2+m_1^2+m_2^2)$



$\Big]$

$+\alpha_3 [\alpha_2\leftrightarrow\alpha_3] \quad -2\alpha_1 m_1^2$



(IBP)

IBP gives a possibility

① to find relations between diagrams and, thus, to work only with some independent set, i.e. with so-called master integrals

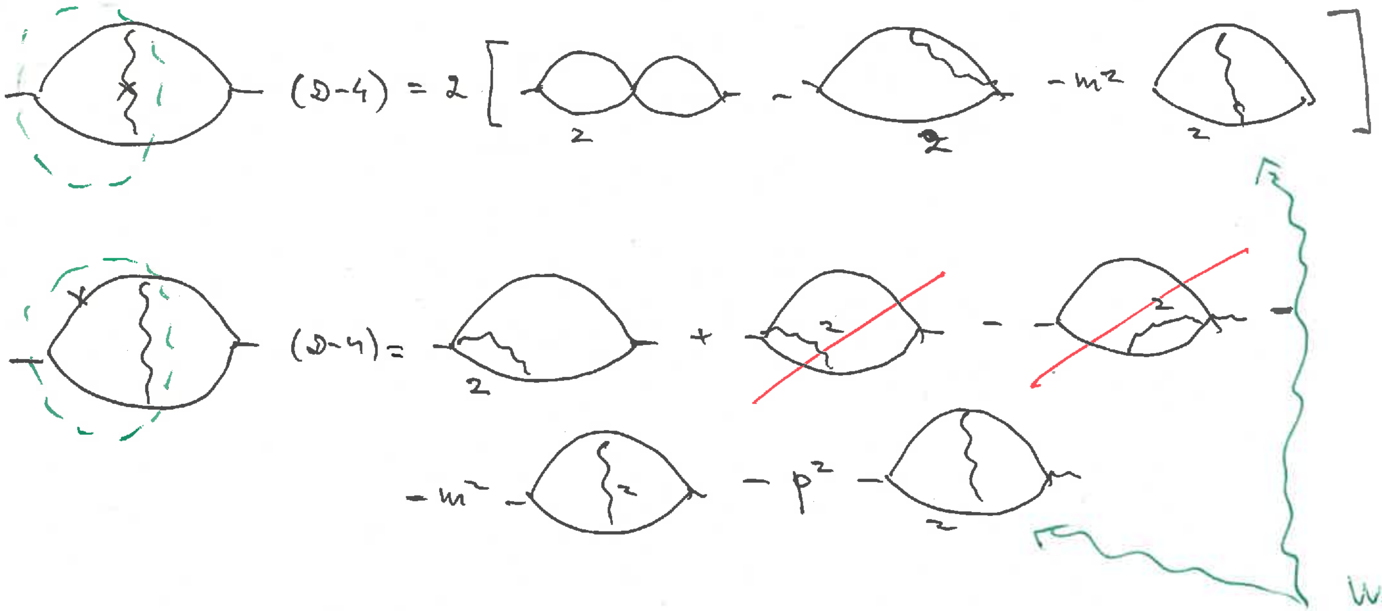
② to calculate master integrals using the differential equations, for example.

Example. The diagram

$= I(m)$

[A.V. Kotikov, 1991]

1 Step.

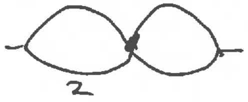


(1)

(2)

We would like to cancel this diagram we can done the following combination: $(1)\cdot p^2 - (2)\cdot 2m^2$

①: $p^2 - 2m^2 (2)$

$I(m) \underbrace{(D-4)}_{-2\varepsilon} [p^2 - 2m^2] = 2 \Big[p^2 \Big\{$ 

$-$ 

$\Big\} - m^2$ 

$- 2m^2 (p^2 - m^2)$ 

$-\frac{\partial}{\partial m^2} I(m)$

So, we have the equation for I(m)

$$\Big[\varepsilon (p^2 - 2m^2) + m^2 (p^2 - m^2) \frac{\partial}{\partial m^2} \Big] I_1(m) = -R$$

$R = p^2 \Big[$ 

$-$ 

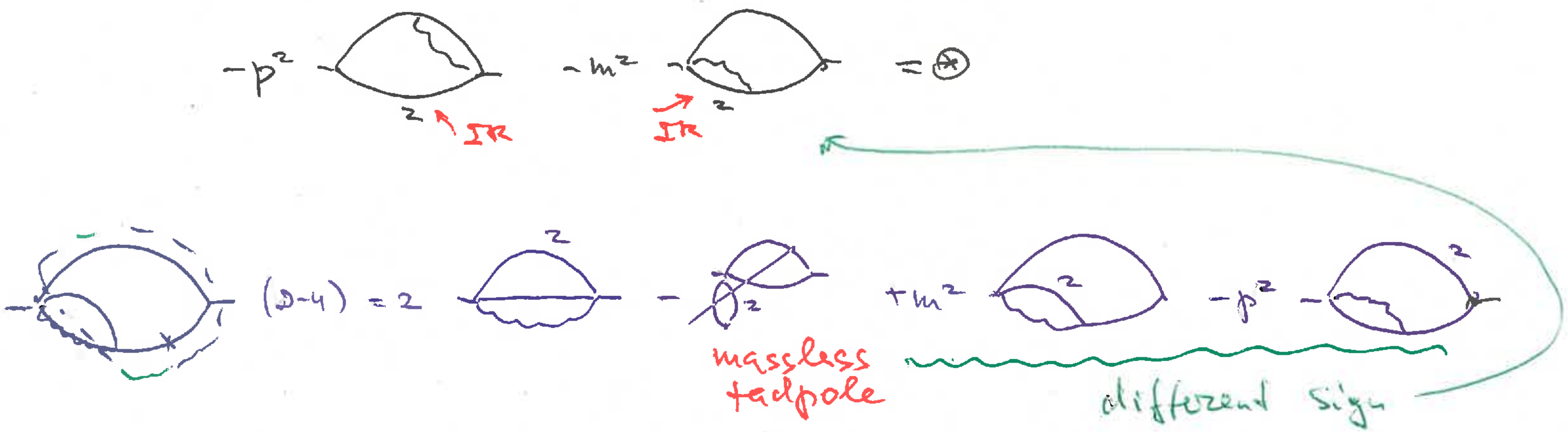
$- \frac{m^2}{p^2}$ 

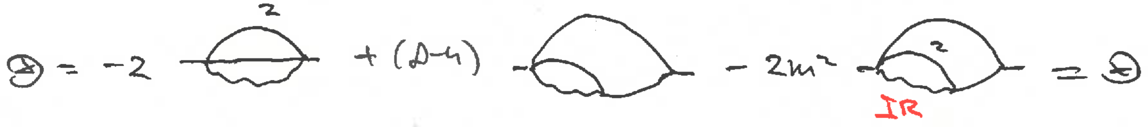
$\Big]$

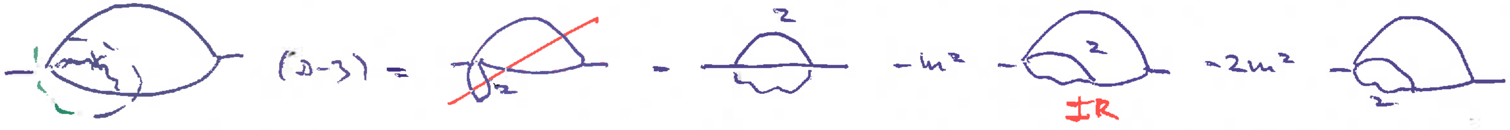
$A(1,1)\, A(2,1) \xrightarrow[1+2\varepsilon]{}$

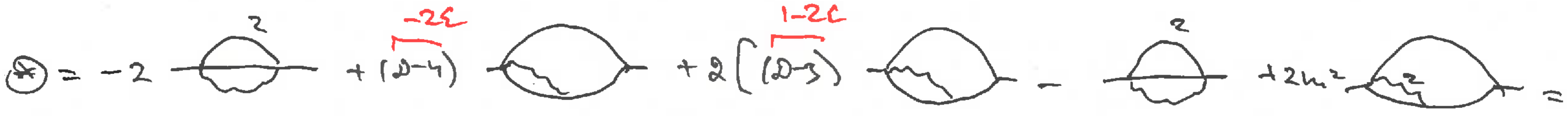
contains only simpler (than I(m)) diagrams [less number of propagators]

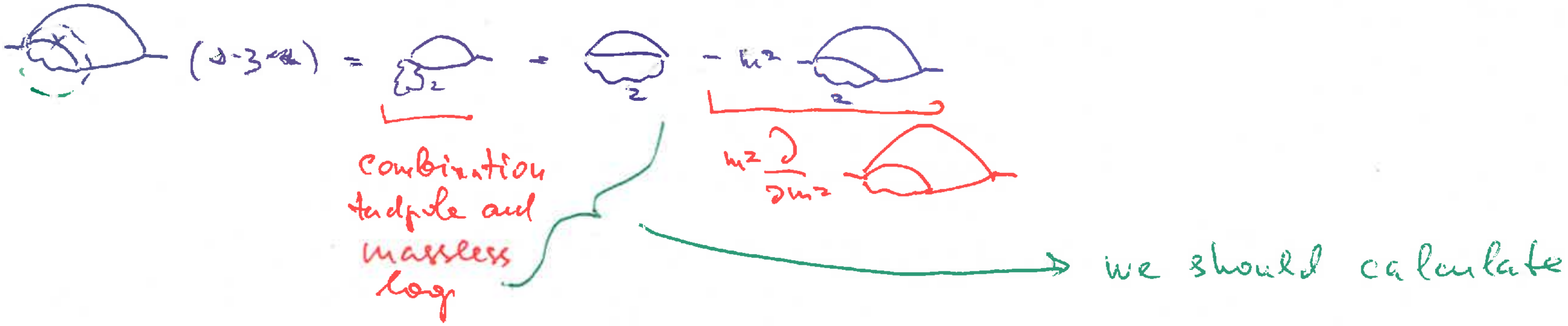
Consider the combination









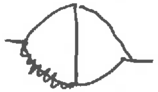


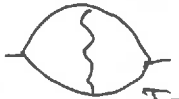
So, we should calculated one loop diagram  (by Feynman parameters, for example) and later to reconstruct the initial diagram I(n) by several integrations

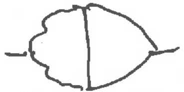
The method is rather powerfull but it needs the work with hands (application of computer is problematic)

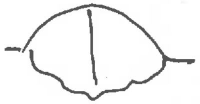
IV Calculation of some FI without "direct calculation".

[J. Fleischer, A.V.K., O.L. Veretin, 1999]


I_1


I_5


I_{12}

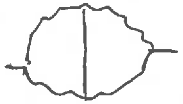

I_{13}


I_{14}

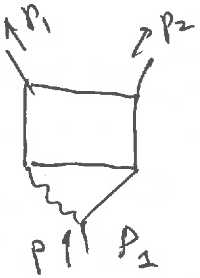

I_{15}


I_{125}

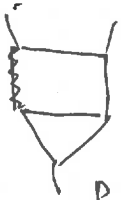
I_{123}

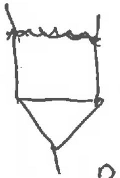

I_{1234}

$p_1^2 = p_2^2 = 0$

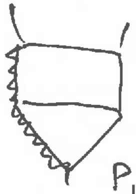


P_1

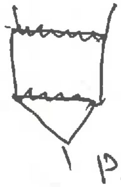

P_3


P_5


P_6


P_{13}


P_{12}


P_{56}


P_{126}

P_{3456}


N_1


N_3


N_{13}

N_{12}

M13

The results have the form: (inverse mass expansion):

$$FI = \frac{\hat{N}}{p^{2a}} \sum_{n=1}^{\infty} c_n (yx)^n \left\{ F_0^{(n)} + \left[\ln(-x)\, F_{1,1}(n) + \frac{1}{\varepsilon} F_{1,2}(n) \right] + \right.$$

$$\left. + \left[\ln^2(-x)\, F_{2,1}(n) + \frac{1}{\varepsilon} \ln(-x)\, F_{2,2}(n) + \frac{1}{\varepsilon^2} F_{2,3}(n) + \zeta(2)\, F_{2,4}(n) \right] + \ldots \right\}$$

$x = \frac{p^2}{m^2}$, $y = \pm$, $a = 1$ for 2-point FI, $a = 2$ for 3-point FI, $\hat{N} = \left(\frac{\mu^2}{m^2}\right)^{2\varepsilon}$

$C_n = \frac{(n!)^2}{(2n)!} \equiv \hat{C}_n$ for FI with two-massive-particle-cuts (2m-cuts)

$C_n = 1$ for [illegible]-cuts.

$$F_{N,k}(n) \approx \frac{S_{\pm a,\ldots}}{n^b}, \frac{\zeta(\pm a)}{n^b}, \underbrace{\frac{V_{a,\ldots}}{n^b}, \frac{W_{a,\ldots}}{n^b}}_{\text{only for 2m-cuts}}$$

$$S_{\pm a, \pm b, \ldots}(j) = \sum_{m=1}^{j} \frac{(\pm)^m}{m^a} S_{\pm b \ldots}(m), \quad \zeta(\pm a, \pm b, \ldots) = \sum_{m=1}^{\infty} \frac{(\pm)^m}{m^a} S_{\pm b, \ldots}(m)$$

$$V_{a,b\ldots}(j) = \sum_{m=1}^{\infty} \frac{\hat{C}_m}{m} S_{b,\ldots}(m), \quad W_{a,b\ldots}(j) = \sum_{m=1}^{\infty} \frac{\hat{C}_m^{-1}}{m} S_{b,\ldots}(m)$$

M14

let $\Phi_{\pm a_1,\dots,\pm a_n} \sim \zeta(\pm a_1,\dots,\pm a_n) \sim \frac{1}{n^a} \quad \left(\sum_1^n a_i = a\right)$

$\uparrow$ transcendental weight (level of complexity)

Then

$$F_{N,k}(n) \sim \frac{1}{n^{3-N}} \quad (N \geq 2) \quad \text{for two-point FI}$$

$$F_{N,n}(n) \sim \frac{1}{n^{4-N}} \quad (N \geq 3) \quad \text{for three-point FI}$$

In „x" space the corresponding property for weights of Polylogarithms

(with argument x, when $c_n = 1$ and y, when $c_n = \hat{c}_n$)

So, if we calculated the expansion of some poles (or the coefficient in the front of logs) we can predict more complicated terms by increasing of level of complexity.

$$y = \frac{1 - \sqrt{x/x-4}}{1 + \sqrt{x/x-4}}$$

The number of terms ("basis") is limited.

Using the calculation of the first coefficient ($n \leq 100$) we can predict exact result (at any „n").

Moreover, if we calculated (in some way) two-point diagrams we can predict the results for ~~calculation~~ three-point FIs.

Indeed,

$$I_1 = \frac{\hat{N}}{q^2} \sum_{n=1} \frac{x^n}{n} \left\{ \frac{1}{2} \ln^2(-x) - \frac{2}{n} \ln(-x) + \zeta(2) - 2\zeta_2 - 2\frac{S_1}{n} + \frac{3}{n^2} \right\}$$

$$P_1 = \frac{\hat{N}}{(q^2)^2} \sum_{n=1}^{\infty} \frac{(x)^n}{n} \left\{ -\frac{1}{2\varepsilon^3} - \frac{1}{\varepsilon^2} S_1 + \frac{1}{\varepsilon}\left[-\frac{1}{2}\zeta_2 + \frac{5}{2} S_2 - \frac{1}{2} S_1^2 + \frac{1}{n^2} - \frac{1}{n}\ln(-x) + \frac{1}{2}\ln^2(-x) \right] \right.$$

$$- \frac{8}{3}\zeta_3 - \zeta_2 S_1 - \frac{\zeta_2}{n} + \frac{8}{3} S_3 + \frac{9}{2} S_1 S_2 + \frac{5}{6} S_1^3 + 4\frac{S_2}{n} + 2\frac{S_1}{n^2} + \frac{3}{n^3} +$$

$$\left. + \left(\zeta_2 - 4S_2 - 2\frac{S_1}{n} - \frac{3}{n^2}\right)\ln(-x) + \left(S_1 + \frac{1}{2n}\right)\ln^2(-x) - \frac{1}{2}\ln^3(-x) \right\}$$

$$I_5 = \frac{\hat{N}}{q^2} \sum_{n=1} \frac{(-x)^n}{n} \left\{ -\ln^2(-x) + \frac{2}{n}\ln(-x) - 2\zeta_2 - 4S_{-2} - \frac{2}{n^2} - \frac{2(-1)^n}{n^2} \right\}$$

$$P_5 = \frac{\hat{N}}{(q^2)^2} \sum_{n=1} \frac{(-x)^n}{n} \left\{ -6\zeta_3 + 2\zeta_2 S_1 + 6S_3 - 2S_1 S_2 + 4\frac{S_2}{n} - \frac{S_1^2}{n} + 2\frac{S_1}{n} + \right.$$

$$\left. + \left(-4S_2 + S_1^2 - 2\frac{S_1}{n}\right)\ln(-x) + S_1 \ln^2(-x) \right\}$$

M16

$$I_{12} = \frac{\hat{N}}{q^2} \sum_{n=1}^{\infty} \frac{x^n}{n^2} \left\{ \frac{1}{n} + \hat{C}_n \left(-2\ln(-x) - 3W_{1,1} + \frac{2}{n} \right) \right\}$$

$$P_{12} = \frac{\hat{N}}{(q^2)^2} \sum_{n=1}^{\infty} \frac{x^n}{n^2} \hat{C}_n \Big[\frac{2}{\varepsilon^2} + \frac{2}{\varepsilon} \left(S_1 - 3W_1 + \frac{1}{n} - \ln(-x) \right) + 12W_2 - 18W_{1,1} - 13S_2 + S_1^2 - 6S_1W_1$$

$$+ \frac{2S_1}{n} + \frac{3}{n^2} - 2\left(S_1 + \frac{1}{n}\right)\ln(-x) + \ln^2(-x) \Big]$$

$$W_{12} = \frac{\hat{N}}{(q^2)^2} \sum_{n=1}^{\infty} \frac{x^n}{n} \hat{C}_n \Big[-\frac{6}{\varepsilon^2} W_1 + \frac{1}{\varepsilon} \left[-4W_2 - 12W_{1,1} - 12S_1W_1 - 16S_2 \right] -$$

$$- 18S_2W_1 - 4W_3 - 8S_3 + 12W_{1,2} - 8W_{2,1} - 12W_{1,(1+1)} \Big]$$

(i.e. no mixture of function with different transcendental wight)

Same level of complexity in F_{NK} corresponds to the special form DE for FI (A.V.K, 2010) (at least for the planar case):

$$\left(x\frac{d}{dx} + K\varepsilon \right) FI = \text{less complicated diagrams } (\equiv FI_1)$$

For the FI_1 we can prepared ~~same~~ similar equation with r.h.s, containg only less^2 complicated diagrams $(\equiv FI_2)$

Now, it is very popular the method to use the matrix homog. equation in the so-called ε-form (J. Henn, 2013)

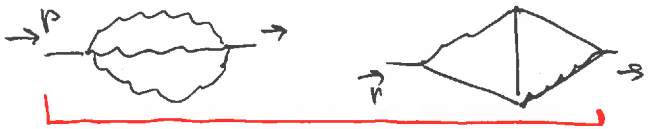
$$\frac{d}{dx}\hat{FI} - \varepsilon\hat{K}\,\hat{FI} = 0 \quad ,$$ where

$$\hat{FI} = \begin{pmatrix} FI \\ FI_1/\varepsilon \\ \vdots \\ FI_n/\varepsilon^n \end{pmatrix}$$

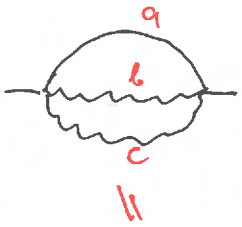
!!! [lot of application in the last years] !!!

V Elliptic results

[Remiddi et al., 2005, 2016, 2017]
[B.A. Kniel et al. 2006]
[L. Adams et al., 2014–2018]
[S. Bloch et al., 2013, 2016]



Simplest 3m-cuts diagrams with Elliptics in their results
[three-massive-particle-cuts (3m-cuts)]

We consider 

$q^2 = m^2$, which is important for non-relativistic QED and QCD, when $M = m$

Min. space

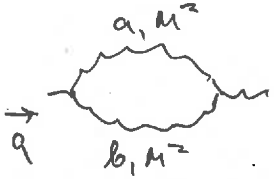
$\| \; J^{mMM}_{abc}$

Here $\text{———} = \dfrac{1}{p^2 - m^2}$

$\sim\!\sim\!\sim = \dfrac{1}{p^2 - M^2}$

Min. space !!

Effective mass method:



$$= i^{1+d} \frac{\Gamma(a+b-D/2)}{\Gamma(a)\Gamma(b)} \int_0^1 \frac{ds}{(1-s)^{a+1-D/2}\, s^{b+1-D/2}} \underset{\mu^2 = \frac{M^2}{s(1-s)}}{\overset{a+b-D/2}{\sim\!\sim\!\sim\!\sim}}$$

M19

Applying DE method to one loop diagrams with the effective mass and ~~remove~~ integrated later on s we have $\left(x = \frac{m^2}{M^2}\right)$

$$M^2 J^{mMM}_{122} = \frac{1}{x} \sum_{n=1}^{\infty} (-x)^n \bar{C}_n \cdot \frac{1}{2n} \left[-\ln x + S_1 - 3\bar{S}_1 + 2\bar{\bar{S}}_1 + \frac{1}{2n^2} \right] +$$

$$+ \frac{1}{x^2} \sum_{n=1}^{\infty} \frac{(-16x^2)^n}{\binom{2n}{n}\binom{4n}{2n}} \cdot \cancel{\tfrac{1}{2}} \left[-\frac{1}{2n-1} + \frac{1}{2n} - \frac{1}{4n^2} \right]$$

$$\bar{C}_n = \frac{\binom{2n}{n}}{\binom{4n}{2n}}$$

$S_1 = S_1(n-1)$

$\bar{S}_1 = S_1(2n-1)$

$\bar{\bar{S}}_1 = S_1(4n-1)$

$$(M^2)^{2\varepsilon} J^{mMM}_{112} = -\frac{1}{2\varepsilon^2} - \frac{1}{\varepsilon}\left(\ln x + \frac{1}{2}\right) - \ln^2 x - \ln x - \frac{1}{2}\zeta_2 - \frac{1}{2}$$

$$+ \frac{1}{x} \sum_{n=1}^{\infty} (-x^2)^n \bar{C}_n \cdot \frac{4n-1}{2n(2n-1)^2} \left[\ln x - S_1 + 3\bar{S}_1 - 2S_1 - \frac{1}{2n} - \frac{2}{2n-1} + \frac{2}{4n-1} \right]$$

$$+ \sum_{n=1}^{\infty} \frac{(-16x^2)^n}{\binom{2n}{n}\binom{4n}{n}} \frac{1}{4n^2(2n+1)}$$

J^{mMM}_{111} has similar structure

Integral representations

$$M^2 J^{mMM}_{122} = -\frac{1}{2x}\int_0^1 \frac{dt}{t\sqrt{1-t}}\left(\frac{1}{\sqrt{1+4A^2}}L(A) - 2\ln A\right)$$

$A = \frac{xt}{4}$, $L(x) = L_1(x) + L_2(x)$

$L_1(x) = \ln\frac{\sqrt{\ }+1}{\sqrt{\ }-1}$, $L_2(x) = \frac{\sqrt{\ }-2x}{\sqrt{\ }+2x}$

$$(M^2)^{2\epsilon} J^{mMM}_{112} = -\frac{1}{2\epsilon^2} - \frac{1}{\epsilon}\left(\ln x - \frac{1}{2}\right) - \ln^2 x - \ln x - \frac{1}{2}\zeta(2) - \frac{1}{2}$$

$$-\frac{1}{x}\int_0^1 \frac{dt}{t^2\sqrt{1-t}}\left(\sqrt{1+4x^2}\,L(A) - 2\ln A + 4A\right)$$

Integral J^{mMM}_{111} has similar results

$\int_0^1 \frac{dt}{t\sqrt{1-t}\sqrt{1+4A^2}}$ is the Elliptic integral of the third kind [EI3K]

So, the results of above integrals are integrals of EI3K.

They should be expressed in the form of Elliptic Polylogarithms.

Now we are working for this subject.

Conclusion

I have considered several approaches to calculate Feynman integrals (some of them are very old)

- Feynman parameterization
- Integration by parts
- Method of Uniqueness + transformation
- Gegenbauer Polynomials
- Differential equations
- method of „effective mass“.

Thank you for your attentions !!!