Hopf algebra for Feynman diagrams and integrals

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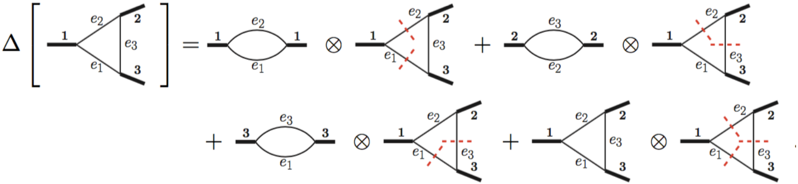
ETH
29 May 2018

Based on 1702.03163, 1703.05064, 1704.07931 with Abreu, Duhr, and Gardi; and work in preparation with Abreu, Duhr, Gardi, and Matthew

Feynman integrals

- Important for detailed analysis of multiloop amplitudes, e.g. for LHC physics discoveries
- Difficult to compute; key tools include differential equations and discontinuities
- Have a Hopf algebra structure that maps the functions to simpler ones in a way that exposes their behavior in differential equations and discontinuities
- At 1 loop, this Hopf algebra is consistent with the Hopf algebra of multiple polylogarithms, in the expansion of Feynman integrals in dimensional regularization

Operations of the Hopf algebra



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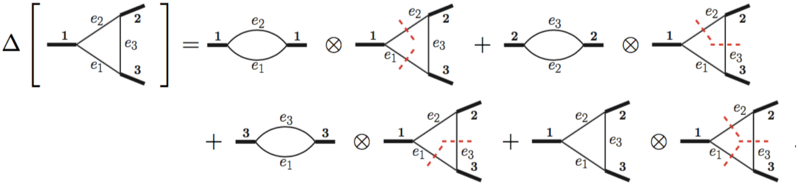


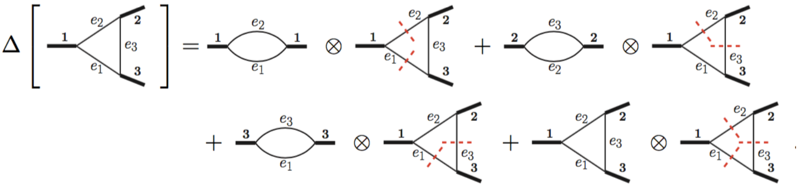
$$\Delta(\log z) = 1 \otimes \log z + \log z \otimes 1$$

$$\Delta(\log^2 z) = 1 \otimes \log^2 z + 2 \log z \otimes \log z + \log^2 z \otimes 1$$

$$\Delta(\mathrm{Li}_2(z)) = 1 \otimes \mathrm{Li}_2(z) + \mathrm{Li}_2(z) \otimes 1 + \mathrm{Li}_1(z) \otimes \log z$$

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt$$

Operations of the Hopf algebra



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\end{aligned}
$$

Discontinuities and cuts:

$$\Delta \; \mathrm{Disc} = (\mathrm{Disc} \otimes 1) \, \Delta$$

Differential operators:

$$\Delta \, \partial = (1 \otimes \partial) \, \Delta$$

Master formula for Hopf algebra on integrals

We conjecture a framework as follows.

Coactions of the following form:

$$\Delta\left(\int_\gamma \omega\right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$$

with a duality condition

$$P_{ss} \int_{\gamma_i} \omega_j = \delta_{ij} \, .$$

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$$P_{ss} \int_{\gamma_i} \omega_j = \delta_{ij}\,.$$

P_{ss} is semi-simple projection ("drop logarithms but not π").

The master formula coaction is like inserting a complete set of states ("ω_i are a set of master integrands for ω").

Outline

Algebras and bialgebras

An algebra H is a ring (addition group & multiplication)
which has a multiplicative unit (1)
and which is also a vector space over a field K.

Example: $n \times n$ matrices with entries in K.
In this talk, the field is always $K = \mathbb{Q}$.

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A bialgebra is an algebra H with two more maps, the coproduct $\Delta : H \to H \otimes H$, and the counit $\varepsilon : H \to \mathbb{Q}$, satisfying the following axioms.

- Coassociativity: $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$
- Δ and ε are algebra homomorphisms:
  $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b) \;\text{ and }\; \varepsilon(a \cdot b) = \varepsilon(a) \cdot \varepsilon(b)$
- The counit and the coproduct are related by $(\varepsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}$

The "incidence" bialgebra [Joni, Rota]

A simple combinatorial algebra: let $[\mathbf{n}] = \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{n}\}$.
Elements: pairs of nested subsets $S \subseteq T$, where $S \subseteq T \subseteq [\mathbf{n}]$.

$\{\mathbf{1}\} \subseteq \{\mathbf{1}, \mathbf{2}\}$ represented by $\mathbf{1}\,\mathbf{2}$
$\emptyset \subseteq \{\mathbf{1}, \mathbf{2}\}$ represented by $\mathbf{1}\,\mathbf{2}$
$\emptyset \subset \emptyset$ represented by $*$

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Multiplication is free, and the coproduct is defined by

$$\Delta(S \subseteq T) = \sum_{S \subseteq X \subseteq T} (S \subseteq X) \otimes (X \subseteq T).$$

For example:

$$\begin{aligned}
\Delta(\mathbf{1}\,\mathbf{2}) &= \mathbf{1}\,\mathbf{2} \otimes \mathbf{1}\,\mathbf{2} + \mathbf{1} \otimes \mathbf{1}\,\mathbf{2} + \mathbf{2} \otimes \mathbf{1}\,\mathbf{2} + * \otimes \mathbf{1}\,\mathbf{2} \\
\Delta(\mathbf{1}\,\mathbf{2}) &= \mathbf{1}\,\mathbf{2} \otimes \mathbf{1}\,\mathbf{2} + \mathbf{2} \otimes \mathbf{1}\,\mathbf{2} \\
\Delta(\mathbf{2}) &= \mathbf{2} \otimes \mathbf{2} + * \otimes \mathbf{2} \\
\Delta(\mathbf{2}) &= \mathbf{2} \otimes \mathbf{2} \\
\Delta(S \subseteq S) &= (S \subseteq S) \otimes (S \subseteq S)
\end{aligned}$$

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Multiplication is free, and the coproduct is defined by

$$\Delta(S \subseteq T) = \sum_{S \subseteq X \subseteq T} (S \subseteq X) \otimes (X \subseteq T).$$

The counit is

$$\varepsilon(S \subseteq T) = \begin{cases} 1\,, & \text{if } S = T\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

e.g. $\varepsilon(\mathbf{2}) = 0, \quad \varepsilon(\mathbf{1}\,\mathbf{2}) = 0, \quad \varepsilon(\mathbf{2}) = 1, \quad \varepsilon(*) = 1$

Illustration of axioms

$$\begin{aligned}
\Delta(\mathbf{1\,2}) &= \mathbf{1\,2} \otimes \mathbf{1\,2} + \mathbf{1} \otimes \mathbf{1\,2} + \mathbf{2} \otimes \mathbf{1\,2} + * \otimes \mathbf{1\,2} \\
\Delta(\mathbf{2}) &= \mathbf{2} \otimes \mathbf{2} + * \otimes \mathbf{2} \\
\Delta(\mathbf{2}) &= \mathbf{2} \otimes \mathbf{2} \\
\Delta(*) &= * \otimes *
\end{aligned}$$

- Coassociativity of the coproduct, $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$

$$\begin{aligned}
(\Delta \otimes \mathrm{id})\Delta(\mathbf{2}) &= \Delta(\mathbf{2}) \otimes \mathbf{2} + \Delta(*) \otimes \mathbf{2} \\
&= \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} + * \otimes \mathbf{2} \otimes \mathbf{2} + * \otimes * \otimes \mathbf{2} \\
&= \mathbf{2} \otimes \Delta(\mathbf{2}) + * \otimes \Delta(\mathbf{2}) \\
&= (\mathrm{id} \otimes \Delta)\Delta(\mathbf{2})
\end{aligned}$$

- Counit, $(\varepsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}$

$$\varepsilon(\mathbf{2}) \otimes \mathbf{2} + \varepsilon(*) \otimes \mathbf{2} = \mathbf{2} \otimes \varepsilon(\mathbf{2}) + * \otimes \varepsilon(\mathbf{2}) = \mathbf{2}$$

Coproduct and coaction

If H is a Hopf algebra, then a H (right-) comodule is a vector space A with a map $\rho : A \to A \otimes H$ such that

$$(\rho \otimes \mathrm{id})\rho = (\mathrm{id} \otimes \Delta)\rho \quad \text{and} \quad (\mathrm{id} \otimes \varepsilon)\rho = \mathrm{id}\,.$$

Here Δ is a coproduct on H. ρ is a coaction on A.

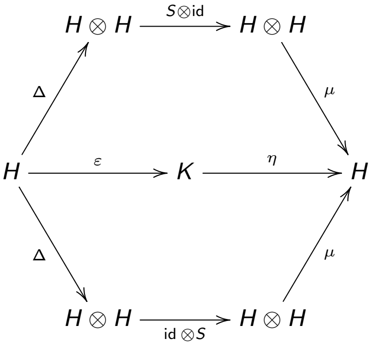
MPLs modulo $i\pi$ form a Hopf algebra H. For the full space of MPLs, we need the comodule $\mathbb{Q}[i\pi] \otimes H$, with a coaction ρ where $\rho(i\pi) = i\pi \otimes 1$.

[Goncharov, Duhr, Brown]

In this presentation, the distinction is not terribly important. We continue to use Δ to denote our operations, which are formally coactions of bialgebras.

Hopf algebras

A Hopf algebra is a bialgebra H with an *antipode* map $S : H \to H$ that satisfies



(Here μ, η denote multiplication and inclusion, respectively.)

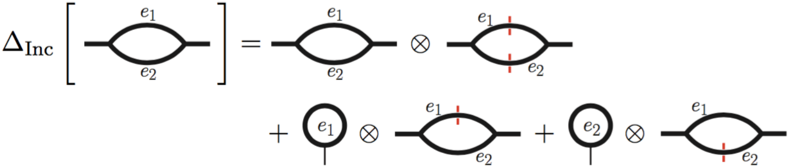
The incidence bialgebra becomes a Hopf algebra if we adjoin inverse elements $(S \subseteq S)^{-1}$.

Example of the incidence algebra: edges of graphs

$$\Delta(1\,2) \quad = \quad 1\,2 \otimes 1\,2 + 1 \otimes 1\,2 + 2 \otimes 1\,2 + * \otimes 1\,2$$

For graphs, set $* = (\emptyset \subseteq \emptyset) = 0$.

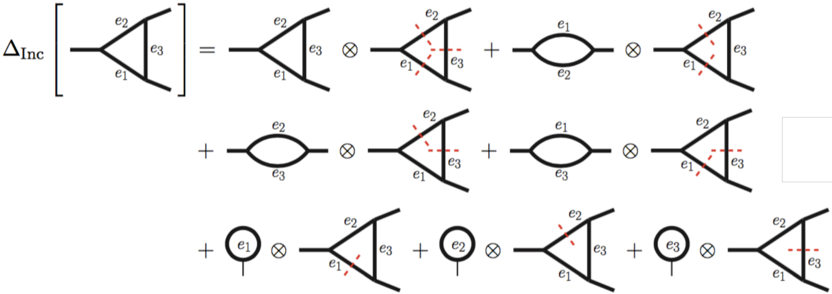
Pinch and cut *complementary* subsets of edges:



Example of the incidence algebra: edges of graphs

$$\begin{aligned}\Delta(1\,2\,3) \;=\;& 1\,2\,3 \otimes 1\,2\,3 + 1\,2 \otimes 1\,2\,3 + 2\,3 \otimes 1\,2\,3 + 1\,3 \otimes 1\,2\,3 \\ & + 1 \otimes 1\,2\,3 + 2 \otimes 1\,2\,3 + 3 \otimes 1\,2\,3 + * \otimes 1\,2\,3\end{aligned}$$

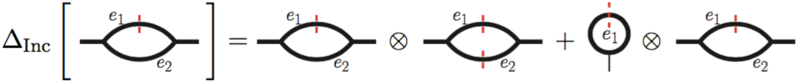
Pinch and cut *complementary* subsets of edges:



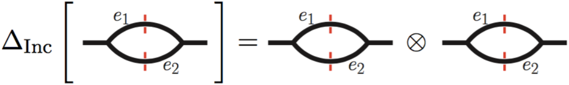
Example of the incidence algebra: edges of graphs

Can also start with a cut diagram.

$$\Delta(1\,2) \quad = \quad 1\,2 \otimes 1\,2 + 1 \otimes 1\,2$$



$$\Delta(1\,2) \quad = \quad 1\,2 \otimes 1\,2$$



Multiple polylogarithms (MPL)

A large class of iterated integrals are described by multiple polylogarithms:

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

Examples:

$$G(0; z) = \log z, \quad G(a; z) = \log\left(1 - \frac{z}{a}\right)$$
$$G(\vec{a}_n; z) = \frac{1}{n!} \log^n\left(1 - \frac{z}{a}\right), \quad G(\vec{0}_{n-1}, a; z) = -\mathrm{Li}_n\left(\frac{z}{a}\right)$$

Harmonic polylog if all $a_i \in \{-1, 0, 1\}$.
n is the *transcendental weight*.

Many Feynman integrals can be written in terms of classical and harmonic polylogs.

Hopf algebra of MPL [Goncharov; Duhr]

Closure under multiplication via the shuffle product:

$$G(\vec{a}_1; z)\, G(\vec{a}_2; z) = \sum_{\vec{a} \in \vec{a}_1 \,\text{Ш}\, \vec{a}_2} G(\vec{a}; z)\,,$$

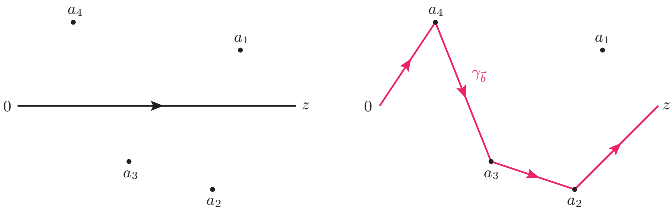
where $\vec{a}_1 \,\text{Ш}\, \vec{a}_2$ are the permutations preserving the relative orderings of $\vec{a}_1$ and $\vec{a}_2$.

There is a coaction on MPLs. It is graded by weight, and thus breaks MPLs into simpler functions (lower weight).

$$\begin{aligned}
\Delta(\log z) &= 1 \otimes \log z + \log z \otimes 1 \\
\Delta(\log x \log y) &= 1 \otimes (\log x \log y) + \log x \otimes \log y + \log y \otimes \log x + (\log x \log y) \otimes 1 \\
\Delta(\mathrm{Li}_2(z)) &= 1 \otimes \mathrm{Li}_2(z) + \mathrm{Li}_2(z) \otimes 1 + \mathrm{Li}_1(z) \otimes \log z
\end{aligned}$$

The coaction is a pairing of contours and integrands. Recalls the incidence algebra.

$$\Delta_{\mathrm{MPL}}(G(\vec{a};z)) = \sum_{\vec{b}\subseteq\vec{a}} G(\vec{b};z)\otimes G_{\vec{b}}(\vec{a};z)$$



(a) (b)

Contour (b) takes a subset of residues in a given order.

Outline

Feynman integrals

A useful basis for all 1-loop integrals:

$$J_n = \frac{ie^{\gamma_E \epsilon}}{\pi^{D_n/2}} \int d^{D_n}k \prod_{j=1}^{n} \frac{1}{(k-q_j)^2 - m_j^2}$$

- k is the loop momentum
- q_j are sums of external momenta, m_j are internal masses
- Dimensions:

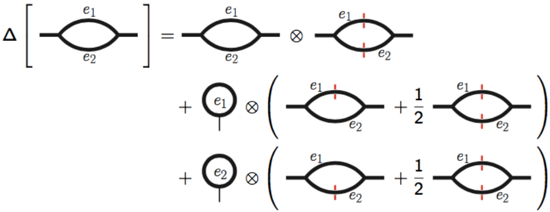
$$D_n = \begin{cases} n - 2\epsilon\,, & \text{for } n \text{ even}\,, \\ n + 1 - 2\epsilon\,, & \text{for } n \text{ odd}\,. \end{cases}$$

  e.g. tadpoles and bubbles in $2 - 2\epsilon$ dimensions,
  triangles and boxes in $4 - 2\epsilon$ dimensions, etc.
- Each J_n has uniform transcendental weight and satisfies nice differential equations.

2 equivalent Hopf algebras

The combinatorial algebra agrees with the Hopf algebra on the MPL of evaluated diagrams!

- The graph with n edges is interpreted as J_n, i.e. in D_n dimensions, no numerator.
- Need to insert extra terms in the diagrammatic equation:



  Isomorphic to the more basic construction. (For any value of 1/2.)

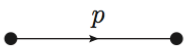
- How do we evaluate the cut graphs?

[related work: Brown; Bloch and Kreimer]

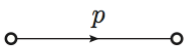
What are generalized cuts?

Traditional [Veltman]:

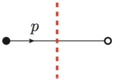
$\bullet = i \qquad \circ = -i$



$= \dfrac{i}{p^2 + i\varepsilon}$



$= \dfrac{-i}{p^2 - i\varepsilon}$



$= 2\pi\,\delta\left(p^2\right)\theta\left(p_0\right)$

Better interpretation: cuts should be understood primarily as residues.

[related: Kosower, Larsen]

- Change the contour, not the integrand
- Consistent with prior expectations
- Fits with diagrammatic coaction

Generalized cuts as residues and determinants

1-loop cuts defined as residues:

$$\mathcal{C}_C[I_n] = \frac{e^{\gamma_E \epsilon}}{i\pi^{\frac{D}{2}}} \int_{\Gamma_C} d^D k \prod_{j \notin C} \frac{1}{(k-q_j)^2 - m_j^2 + i0} \quad \text{mod } i\pi \,,$$

- C is the set of cut propagators
- Contour Γ_C encircles poles of cut propagators

Cut integrals give discontinuities of their uncut counterparts.

Generalized cuts as residues and determinants

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Cut integrals give discontinuities of their uncut counterparts.
Some results:

$$\mathcal{C}_C I_n = \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_C}} \left(\frac{Y_C}{G_C}\right)^{(D-c)/2} \int \frac{d\Omega_{D-c+1}}{i\pi^{D/2}} \left[\prod_{j \notin C} \frac{1}{(k-q_j)^2 - m_j^2}\right]_C$$

where we often find Gram and modified Cayley determinants:

$$\begin{aligned} G_C &= \det\left(q_i \cdot q_j\right)_{i,j \in C \setminus *} \\ Y_C &= \det\left(\frac{1}{2}(m_i^2 + m_j^2 + (q_i - q_j)^2)\right)_{i,j \in C} \end{aligned}$$

Maximal and next-to-maximal cuts

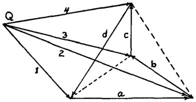
Some special cases:

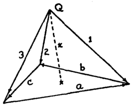
$$
\begin{aligned}
\mathcal{C}_{2k}[J_{2k}] &= \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_{2k}}} \left(\frac{Y_{2k}}{G_{2k}}\right)^{-\epsilon} \\
\mathcal{C}_{2k+1}[J_{2k+1}] &= \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)\sqrt{G_{2k+1}}} \left(\frac{Y_{2k+1}}{G_{2k+1}}\right)^{-\epsilon} .
\end{aligned}
$$

$$
\mathcal{C}_{2k-1}[J_{2k}] = -\frac{\Gamma(1-2\epsilon)^2}{\Gamma(1-\epsilon)^3} \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_{2k}}} \left(\frac{Y_{2k-1}}{G_{2k-1}}\right)^{-\epsilon} {}_2F_1\left(\frac{1}{2}, -\epsilon; 1-\epsilon; \frac{G_{2k}\, Y_{2k-1}}{Y_{2k}\, G_{2k-1}}\right) .
$$

For more complicated cuts, we set up a Feynman parametrization.

Landau conditions are expressed in polytope geometry: these determinants are volumes of simplices. [Cutkosky]





Landau conditions

$$\alpha_i \left[(k^E - q_i^E)^2 + m_i^2 \right] = 0, \quad \forall i\,.$$
$$\sum_{i=1}^{n} \alpha_i (k^E - q_i^E) = 0\,.$$

Therefore

$$\begin{pmatrix} (k^E - q_1^E)\cdot(k^E - q_1^E) & \dots & (k^E - q_1^E)\cdot(k^E - q_c^E) \\ \vdots & \ddots & \vdots \\ (k^E - q_c^E)\cdot(k^E - q_1^E) & \dots & (k^E - q_c^E)\cdot(k^E - q_c^E) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_c \end{pmatrix} = 0\,.$$

Nontrivial solution $\Leftrightarrow$ $Y_C = 0$ and integral over Γ_C: Landau singularities of the first type.

Second-type singularities come from $G_C = 0$ and integral over $\Gamma_{C\cup\infty}$. Contour is pinched at infinity.

Homology theory for Feynman contours

Homology describes the inequivalent integration contours. Also explains why cuts are discontinuities. Use Leray residues. [Fotiadi, Pham; Hwa, Teplitz; Federbusch; Eden, Fairlie, Landshoff, Nuttall, Olive, Polkinghorne,...]

Residues: if

$$\omega = \frac{ds}{s} \wedge \psi + \theta$$

and $S = \{s = 0\}$, while ψ, θ are regular on S, then

$$\mathrm{Res}_S[\omega] = \psi|_S \,.$$

Cut integrals: if $I_n = \int \omega_n$, then

$$\begin{aligned} \mathcal{C}_C[I_n] &= \int_{S_C} \mathrm{Res}_{S_C}[\omega_n] \\ &= (2\pi i)^{-k} \int_{\delta S_C} \omega_n \end{aligned}$$

where δ constructs a “tubular neighborhood” around $S_C = \cap_{i \in C} S_i$, the spherical locus of the cut conditions.

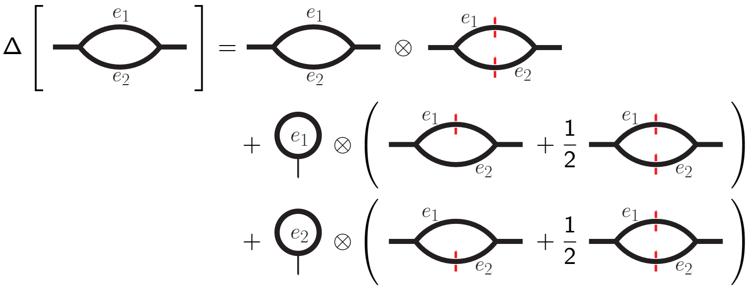
Homology theory for Feynman contours

For 1-loop Feynman integrals, the Decomposition Theorem shows that the contours $\Gamma_C = \delta S_C$ form a basis. [Fotiadi, Pham]

$$\Gamma_{\infty C} = -2x_C\, \Gamma_C - \sum_{C \subset X \subseteq [n]} (-1)^{\lceil \frac{|C|}{2} \rceil + \lceil \frac{|X|}{2} \rceil}\, \Gamma_X\,, \qquad x_c = \begin{cases} 1\,, & |C| \text{ odd}\,, \\ 0\,, & |C| \text{ even}\,. \end{cases}$$

Can work in a compactified projective space, where ∞ is on the same footing as other labels.

Examples of the graphical conjecture

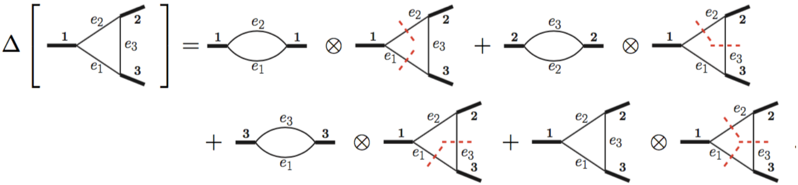


$$\Delta\left(\int_{\Gamma_\emptyset}\omega_{12}\right) = \int_{\Gamma_\emptyset}\omega_{12}\otimes\int_{\Gamma_{12}}\omega_{12} + \int\omega_1\otimes\left(\int_{\Gamma_1}\omega_{12} + \frac{1}{2}\int_{\Gamma_{12}}\omega_{12}\right) + \cdots$$

$$= \int_{\Gamma_\emptyset}\omega_{12}\otimes\int_{\Gamma_{12}}\omega_{12} + \int_{\Gamma_\emptyset}\omega_1\otimes\int_{-\frac{1}{2}\Gamma_{1\infty}}\omega_{12} + \int_{\Gamma_\emptyset}\omega_2\otimes\int_{-\frac{1}{2}\Gamma_{2\infty}}\omega_{12}$$

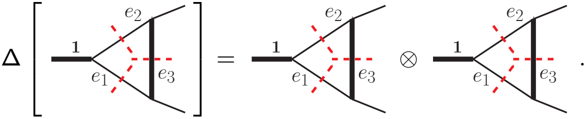
The odd-shaped integrals have poles at infinity.

Examples of the graphical conjecture



Terms with 1/2 are always present in principle, but vanished here due to massless propagators.

Examples of the graphical conjecture



Statement of the graphical conjecture

The coaction on 1-loop graphs defined by pinching and cutting subsets of propagators,

when evaluated by Feynman rules,
if expanded order by order in ϵ,

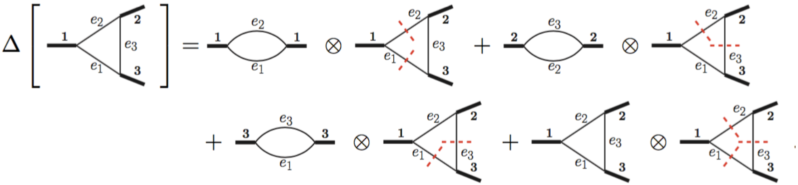
is consistent with the coaction on MPLs!

Evidence for the graphical conjecture

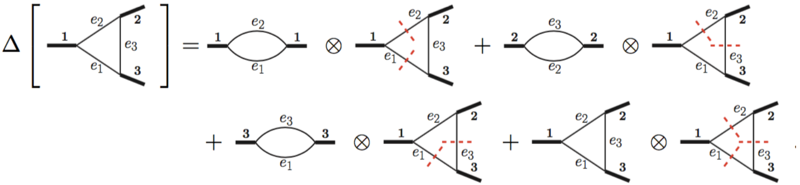
- all tadpoles and bubbles
- triangles and boxes with several combinations of internal and external masses
- consistency checks for more complicated boxes, 0m pentagon, 0m hexagon
- diagrammatic groupings emerging in 2-loop integrals

Checked to several orders in ϵ, or for closed forms with hypergeometric functions.

Coproducts of diagrams



Second entries are discontinuities; first entries have discontinuities.



Second entries are discontinuities; first entries have discontinuities.

Motivated by the identity

$$\Delta\ \mathrm{Disc} = (\mathrm{Disc} \otimes 1)\, \Delta.$$

The companion relation

$$\Delta\, \partial = (1 \otimes \partial)\, \Delta$$

produces differential equations.

Application: cuts and discontinuities

$$\Delta \text{ Disc} = (\text{Disc} \otimes 1)\, \Delta$$

$$\Delta\, (\text{Disc}\, I_n) = (\text{Disc} \otimes 1)\ (\Delta\, I_n)$$

Since $\Delta\,(\text{Disc}\, I_n) = 1 \otimes (\text{Disc}\, I_n) + \cdots$, it is enough to look at the terms $\Delta_{1,w-1} I_n$.

The basis integrals of weight 1 are precisely the tadpoles and bubbles. The corresponding cut diagrams have 1 or 2 propagators cut.

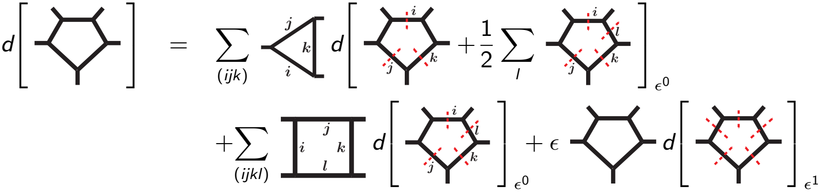
Therefore: the discontinuities are precisely the unitarity cut diagrams (momentum invariant discontinuities) and the single-cut diagrams (mass discontinuities).

Generalized cuts can be interpreted as well.

Application: differential equations

$$\Delta\,\partial = (1 \otimes \partial)\,\Delta$$

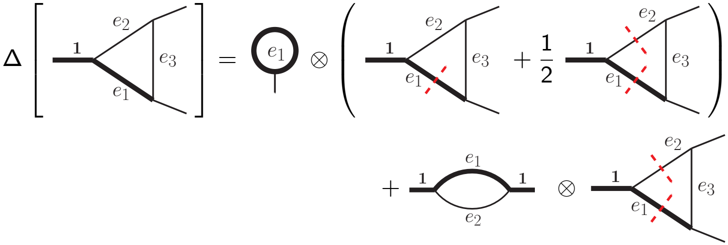
Likewise, we get differential equations by focusing on nearly-maximal cuts in the second factor:



This also shows a way to identify the symbol alphabet.

Outline

Consider the diagrammatic coaction



There is a coaction on $_2F_1$ that gives

$$\begin{aligned}\Delta\,{}_2F_1\left(1,1+\epsilon,2-\epsilon,x\right) &= {}_2F_1\left(1,\epsilon,1-\epsilon,x\right)\otimes{}_2F_1\left(1,1+\epsilon,2-\epsilon,x\right)\\ &\quad+{}_2F_1\left(1,1+\epsilon,2-\epsilon,x\right)\otimes{}_2F_1\left(1,\epsilon,1-\epsilon,\frac{1}{x}\right)\end{aligned}$$

without expanding in ϵ!

Master formula for Hopf algebra on integrals

Coaction of the form

$$\Delta \left(\int_\gamma \omega \right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$$

with a duality condition

$$P_{ss} \int_{\gamma_i} \omega_j = \delta_{ij} \, .$$

P_{ss} is semi-simple projection ("drop logarithms but not π").

To be precise, P_{ss} projects onto the space of semi-simple numbers x satisfying $\Delta(x) = x \otimes 1$.

The master formula for the ${}_2F_1$ family

Consider the family of integrands

$$\omega(\alpha_1, \alpha_2, \alpha_3) = x^{\alpha_1}(1-x)^{\alpha_2}(1-zx)^{\alpha_3}\ dx$$

where $\alpha_i = n_i + \epsilon_i$ and $n_i \in \mathbb{Z}$.

$$\int_0^1 \omega(\alpha_1, \alpha_2, \alpha_3) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2 - \alpha_1)}{\Gamma(\alpha_2)}\ {}_2F_1(-\alpha_3, \alpha_1 + 1; \alpha_2 + \alpha_1 + 2; z)$$

Basis of master integrands:

$$\int_0^1 \omega = c_0 \int_0^1 \omega_0 + c_1 \int_0^1 \omega_1$$

where

$$\begin{aligned} \omega_0 &= x^{\epsilon_1}(1-x)^{-1+\epsilon_2}(1-zx)^{\epsilon_3} \\ \omega_1 &= x^{\epsilon_1}(1-x)^{\epsilon_2}(1-zx)^{-1+\epsilon_3} \end{aligned}$$

With the two contours $\gamma_0 = [0,1]$ and $\gamma_1 = [0, 1/z]$, we have $P_{ss} \int_{\gamma_i} \omega_j \sim \delta_{ij}$.

Master formula for Appell F_1

Family of integrands for F_1.

$$\omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \quad = \quad x^{\alpha_1}(1-x)^{\alpha_2}(1-z_1x)^{\alpha_3}(1-z_2x)^{\alpha_4}\,dx$$

where $\alpha_i = n_i + \epsilon_i$ and $n_i \in \mathbb{Z}$.

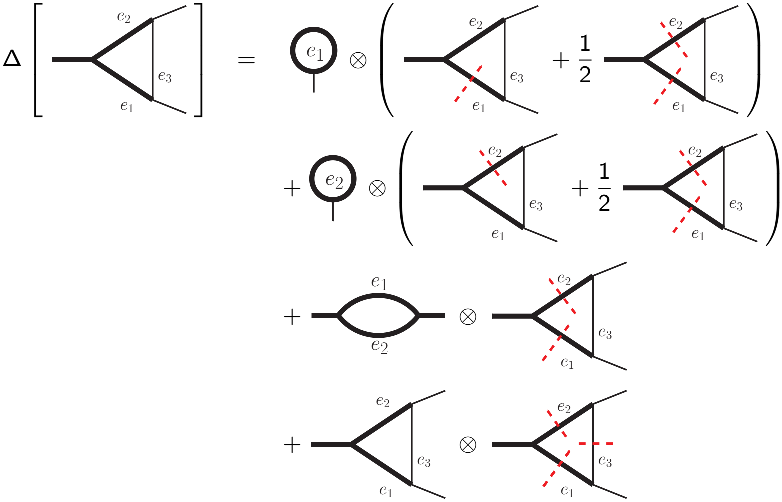
$$\int_0^1 \omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2-\alpha_1)}{\Gamma(\alpha_2)}\, F_1(\alpha_1, \alpha_3, \alpha_4, \alpha_2; z_1, z_2)$$

Master integrands:

$$\begin{aligned}
\omega_0 &= x^{\epsilon_1}(1-x)^{-1+\epsilon_2}(1-z_1x)^{\epsilon_3}(1-z_2x)^{\epsilon_4} \\
\omega_1 &= x^{\epsilon_1}(1-x)^{\epsilon_2}(1-z_1x)^{-1+\epsilon_3}(1-z_2x)^{\epsilon_4} \\
\omega_2 &= x^{\epsilon_1}(1-x)^{\epsilon_2}(1-z_1x)^{\epsilon_3}(1-z_2x)^{-1+\epsilon_4}
\end{aligned}$$

Master contours: $\gamma_0 = [0,1]$, $\gamma_1 = [0, z_1^{-1}]$, $\gamma_2 = [0, z_2^{-1}]$.

Diagrammatic example with F_1



Master formula for ${}_{p+1}F_p$

Family of integrands for ${}_3F_2$.

$$\omega(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = x^{\alpha_1}(1-x)^{\alpha_2}y^{\alpha_3}(1-y)^{\alpha_4}(1-zxy)^{\alpha_5}\,dx\,dy$$

where $\alpha_i = n_i + \epsilon_i$ and $n_i \in \mathbb{Z}$.
Then

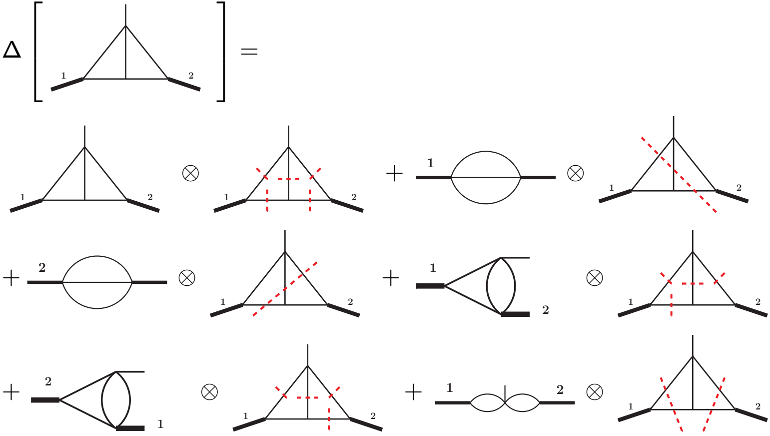
$$\int_0^1\int_0^1 \omega(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \frac{\Gamma()\Gamma()\Gamma()\Gamma()}{\Gamma()\Gamma()}\,{}_3F_2(\alpha_1+1,\alpha_3+1,-\alpha_5;2+\alpha_1+\alpha_2,2+\alpha_3+\alpha_4;z)$$

Basis of master integrands:

$$\begin{aligned}
\omega_0 &= x^{\epsilon_1}(1-x)^{-1+\epsilon_2}y^{\epsilon_3}(1-y)^{-1+\epsilon_4}(1-zxy)^{\epsilon_5}\\
\omega_1 &= x^{\epsilon_1}(1-x)^{-1+\epsilon_2}y^{\epsilon_3}(1-y)^{\epsilon_4}(1-zxy)^{-1+\epsilon_5}\\
\omega_2 &= x^{\epsilon_1}(1-x)^{\epsilon_2}y^{\epsilon_3}(1-y)^{-1+\epsilon_4}(1-zxy)^{-1+\epsilon_5}
\end{aligned}$$

With the master contours $\gamma_0 = \int_0^1 dx \int_0^1 dy$, $\gamma_1 = \int_0^1 dx \int_0^{1/zx} dy$, $\gamma_1 = \int_0^1 dy \int_0^{1/zy} dx$, we find that $P_{ss}\int_{\gamma_i}\omega_j \sim \delta_{ij}$

Diagrammatic example with ${}_3F_2$



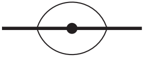
(with various prefactors and dimension shifts inserted to produce pure integrals)

Features of diagrammatic coaction at two loops

Matrix of integrands and contours for *each topology*.

Example: sunrise with one internal mass. 2 master integrands in top topology.

$$= \int_{\Gamma_\emptyset} \omega_{111} \sim {}_2F_1\left(1+2\epsilon, 1+\epsilon, 1-\epsilon, p^2/m^2\right)$$



$$= \int_{\Gamma_\emptyset} \omega_{121} \sim {}_2F_1\left(2+2\epsilon, 1+\epsilon, 1-\epsilon, p^2/m^2\right)$$

For each, only two of the generalized cuts are linearly independent!

Thus 2 independent integration contours, e.g. $\Gamma_\emptyset$ and Γ_{123}.

Diagonalize the matrix: $\int_{\gamma_i} \omega_j \sim \delta_{ij}$ with

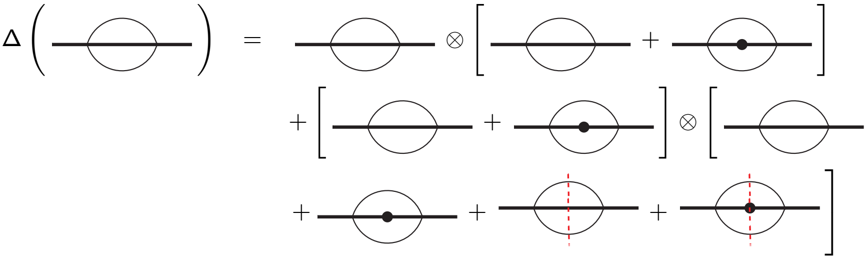
$$\omega_1 = a\epsilon^2 \omega_{111}, \qquad \omega_2 = b\epsilon\omega_{111} + c\epsilon\omega_{121}$$

$$\gamma_1 = \Gamma_\emptyset, \qquad \gamma_2 = -\frac{1}{6\epsilon}\Gamma_{123} + \frac{2}{3}\Gamma_\emptyset$$

Coaction $\Delta\left(\int_\gamma \omega\right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$ is expressible in terms of diagrams.

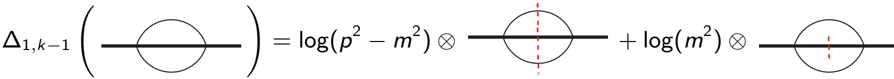
Features of diagrammatic coaction at two loops

For example:



(with prefactors as seen on previous slide)

In particular, we can recover weight 1 discontinuities:



Summary & Outlook

- We observe a Hopf algebra structure on Feynman diagrams. At 1 loop, there is a basis for which the coaction is simply related to pinches and cuts of the original diagram. Beyond 1-loop: encounter matrix equations (cf. higher-order differential equations)

- Corresponds to Goncharov's Hopf algebra on MPLs, with prospects for extensions to hypergeometric integrals and beyond.

- Cuts should be understood through homology and Leray residues.

- Deep connections to discontinuities and differential equations, which are tools for computation.

- Abstracted master formula: a Hopf algebra based on matched pairs of integrands and contours

- To explore further: systematic description beyond 1 loop, full range of hypergeometric functions, applications to integral and amplitude computations.