

Hopf algebra for Feynman diagrams and integrals

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Feynman integrals

- Important for detailed analysis of multiloop amplitudes, e.g. for LHC physics discoveries
- Difficult to compute; key tools include [differential equations](#) and [discontinuities](#)
- Have a [Hopf algebra structure](#) that maps the functions to simpler ones in a way that [exposes](#) their behavior in differential equations and discontinuities
- At 1 loop, this Hopf algebra is consistent with the Hopf algebra of [multiple polylogarithms](#), in the expansion of Feynman integrals in dimensional regularization

Operations of the Hopf algebra

$$\Delta \left[\text{Diagram} \right] = \text{Diagram}_1 \otimes \text{Diagram}_2 + \text{Diagram}_3 \otimes \text{Diagram}_4 + \text{Diagram}_5 \otimes \text{Diagram}_6 + \text{Diagram}_7 \otimes \text{Diagram}_8$$

The diagram on the left is a triangle with a horizontal line labeled '1' on the left, a top line labeled '2' on the right, and a bottom line labeled '3' on the right. The left edge is labeled 'e1', the top edge 'e2', and the right edge 'e3'.

The first row of the expansion consists of:

- A diagram with a horizontal line labeled '1' on the left and '1' on the right, with a top arc labeled 'e2' and a bottom arc labeled 'e1'.
- A tensor product symbol \otimes .
- A diagram with a horizontal line labeled '1' on the left and '1' on the right, with a top line labeled '2' on the right and a bottom line labeled '3' on the right. The left edge is labeled 'e1'. Dashed red lines represent the edges 'e2' and 'e3'.
- A plus sign $+$.
- A diagram with a horizontal line labeled '2' on the left and '2' on the right, with a top arc labeled 'e3' and a bottom arc labeled 'e2'.
- A tensor product symbol \otimes .
- A diagram with a horizontal line labeled '1' on the left and '1' on the right, with a top line labeled '2' on the right and a bottom line labeled '3' on the right. The left edge is labeled 'e1'. Dashed red lines represent the edges 'e2' and 'e3'.

The second row of the expansion consists of:

- A plus sign $+$.
- A diagram with a horizontal line labeled '3' on the left and '3' on the right, with a top arc labeled 'e3' and a bottom arc labeled 'e1'.
- A tensor product symbol \otimes .
- A diagram with a horizontal line labeled '1' on the left and '1' on the right, with a top line labeled '2' on the right and a bottom line labeled '3' on the right. The left edge is labeled 'e1'. Dashed red lines represent the edges 'e2' and 'e3'.
- A plus sign $+$.
- A diagram with a horizontal line labeled '1' on the left and '1' on the right, with a top line labeled '2' on the right and a bottom line labeled '3' on the right. The left edge is labeled 'e1'. Dashed red lines represent the edges 'e2' and 'e3'.
- A tensor product symbol \otimes .
- A diagram with a horizontal line labeled '1' on the left and '1' on the right, with a top line labeled '2' on the right and a bottom line labeled '3' on the right. The left edge is labeled 'e1'. Dashed red lines represent the edges 'e2' and 'e3'.

Operations of the Hopf algebra

$$\Delta \left[\text{Diagram} \right] = \text{Diagram 1} \otimes \text{Diagram 2} + \text{Diagram 3} \otimes \text{Diagram 4} + \text{Diagram 5} \otimes \text{Diagram 6} + \text{Diagram 7} \otimes \text{Diagram 8}$$

The diagram on the left is a vertex with three incoming lines labeled e_1 , e_2 , and e_3 from the left, and three outgoing lines labeled 2, 2, and 3 from the right. The four diagrams in the first row are: 1) a loop with e_2 on top and e_1 on bottom, and 1 on both ends; 2) a vertex with e_2 on top, e_3 on middle, and e_1 on bottom, with dashed red lines on e_2 and e_1 ; 3) a loop with e_3 on top and e_2 on bottom, and 2 on both ends; 4) a vertex with e_2 on top, e_3 on middle, and e_1 on bottom, with dashed red lines on e_2 and e_3 . The four diagrams in the second row are: 5) a loop with e_3 on top and e_1 on bottom, and 3 on both ends; 6) a vertex with e_2 on top, e_3 on middle, and e_1 on bottom, with dashed red lines on e_1 and e_3 ; 7) a vertex with e_2 on top, e_3 on middle, and e_1 on bottom; 8) a vertex with e_2 on top, e_3 on middle, and e_1 on bottom, with dashed red lines on e_2 and e_1 .

$$\Delta(\log z) = 1 \otimes \log z + \log z \otimes 1$$

$$\Delta(\log^2 z) = 1 \otimes \log^2 z + 2 \log z \otimes \log z + \log^2 z \otimes 1$$

$$\Delta(\text{Li}_2(z)) = 1 \otimes \text{Li}_2(z) + \text{Li}_2(z) \otimes 1 + \text{Li}_1(z) \otimes \log z$$

$$\text{Li}_2(z) = - \int_0^z \frac{\log(1-t)}{t} dt$$

Operations of the Hopf algebra

$$\Delta \left[\text{Diagram} \right] = \text{Diagram}_1 \otimes \text{Diagram}_2 + \text{Diagram}_3 \otimes \text{Diagram}_4 + \text{Diagram}_5 \otimes \text{Diagram}_6 + \text{Diagram}_7 \otimes \text{Diagram}_8$$

The diagram on the left is a vertex with three incoming lines labeled e_1, e_2, e_3 and three outgoing lines labeled $1, 2, 3$. The four terms on the right represent the coproduct of this vertex, where the lines are either solid or dashed red, and the vertices are either 1 or 2 .

$$\begin{aligned} \Delta(\log z) &= 1 \otimes \log z + \log z \otimes 1 \\ \Delta(\log^2 z) &= 1 \otimes \log^2 z + 2 \log z \otimes \log z + \log^2 z \otimes 1 \\ \Delta(\text{Li}_2(z)) &= 1 \otimes \text{Li}_2(z) + \text{Li}_2(z) \otimes 1 + \text{Li}_1(z) \otimes \log z \end{aligned}$$

Discontinuities and cuts:

$$\Delta \text{ Disc} = (\text{Disc} \otimes 1) \Delta$$

Differential operators:

$$\Delta \partial = (1 \otimes \partial) \Delta$$

Master formula for Hopf algebra on integrals

We conjecture a framework as follows.

Coactions of the following form:

$$\Delta \left(\int_{\gamma} \omega \right) = \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \omega$$

with a duality condition

$$P_{ss} \int_{\gamma_i} \omega_j = \delta_{ij}.$$

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with a duality condition

$$P_{ss} \int_{\gamma_i} \omega_j = \delta_{ij}.$$

P_{ss} is semi-simple projection (“drop logarithms but not π ”).

The master formula coaction is like inserting a complete set of states (“ ω_i are a set of master integrands for ω ”).

An **algebra** H is a ring (addition group & multiplication) which has a multiplicative unit (1) and which is also a vector space over a field K .

Example: $n \times n$ matrices with entries in K .
In this talk, the field is always $K = \mathbb{Q}$.

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A **bialgebra** is an algebra H with two more maps, the **coproduct** $\Delta : H \rightarrow H \otimes H$, and the counit $\varepsilon : H \rightarrow \mathbb{Q}$, satisfying the following axioms.

- Coassociativity: $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$
- Δ and ε are algebra homomorphisms:
 $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$ and $\varepsilon(a \cdot b) = \varepsilon(a) \cdot \varepsilon(b)$
- The counit and the coproduct are related by $(\varepsilon \otimes \text{id})\Delta = (\text{id} \otimes \varepsilon)\Delta = \text{id}$

The “incidence” bialgebra [Joni, Rota]

A simple combinatorial algebra: let $[n] = \{1, 2, \dots, n\}$.

Elements: pairs of nested subsets $S \subseteq T$, where $S \subseteq T \subseteq [n]$.

$\{1\} \subseteq \{1, 2\}$ represented by **1 2**

$\emptyset \subseteq \{1, 2\}$ represented by **1 2**

$\emptyset \subset \emptyset$ represented by *

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$\emptyset \subset \emptyset$ represented by *

Multiplication is free, and the coproduct is defined by

$$\Delta(S \subseteq T) = \sum_{S \subseteq X \subseteq T} (S \subseteq X) \otimes (X \subseteq T).$$

For example:

$$\Delta(\mathbf{12}) = \mathbf{12} \otimes \mathbf{12} + \mathbf{1} \otimes \mathbf{12} + \mathbf{2} \otimes \mathbf{12} + * \otimes \mathbf{12}$$

$$\Delta(\mathbf{12}) = \mathbf{12} \otimes \mathbf{12} + \mathbf{2} \otimes \mathbf{12}$$

$$\Delta(\mathbf{2}) = \mathbf{2} \otimes \mathbf{2} + * \otimes \mathbf{2}$$

$$\Delta(\mathbf{2}) = \mathbf{2} \otimes \mathbf{2}$$

$$\Delta(S \subseteq S) = (S \subseteq S) \otimes (S \subseteq S)$$

The “incidence” bialgebra [Joni, Rota]

A simple combinatorial algebra: let $[\mathbf{n}] = \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{n}\}$.

Elements: pairs of nested subsets $S \subseteq T$, where $S \subseteq T \subseteq [\mathbf{n}]$.

$\{\mathbf{1}\} \subseteq \{\mathbf{1}, \mathbf{2}\}$ represented by $\mathbf{12}$

$\emptyset \subseteq \{\mathbf{1}, \mathbf{2}\}$ represented by $\mathbf{12}$

$\emptyset \subset \emptyset$ represented by $*$

Multiplication is free, and the coproduct is defined by

$$\Delta(S \subseteq T) = \sum_{S \subseteq X \subseteq T} (S \subseteq X) \otimes (X \subseteq T).$$

The counit is

$$\varepsilon(S \subseteq T) = \begin{cases} 1, & \text{if } S = T, \\ 0, & \text{otherwise.} \end{cases}$$

e.g. $\varepsilon(\mathbf{2}) = 0$, $\varepsilon(\mathbf{12}) = 0$, $\varepsilon(\mathbf{2}) = 1$, $\varepsilon(*) = 1$

$$\Delta(\mathbf{12}) = \mathbf{12} \otimes \mathbf{12} + \mathbf{1} \otimes \mathbf{12} + \mathbf{2} \otimes \mathbf{12} + * \otimes \mathbf{12}$$

$$\Delta(\mathbf{2}) = \mathbf{2} \otimes \mathbf{2} + * \otimes \mathbf{2}$$

$$\Delta(\mathbf{2}) = \mathbf{2} \otimes \mathbf{2}$$

$$\Delta(*) = * \otimes *$$

- Coassociativity of the coproduct, $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$

$$\begin{aligned} (\Delta \otimes \text{id})\Delta(\mathbf{2}) &= \Delta(\mathbf{2}) \otimes \mathbf{2} + \Delta(*) \otimes \mathbf{2} \\ &= \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} + * \otimes \mathbf{2} \otimes \mathbf{2} + * \otimes * \otimes \mathbf{2} \\ &= \mathbf{2} \otimes \Delta(\mathbf{2}) + * \otimes \Delta(\mathbf{2}) \\ &= (\text{id} \otimes \Delta)\Delta(\mathbf{2}) \end{aligned}$$

- Counit, $(\varepsilon \otimes \text{id})\Delta = (\text{id} \otimes \varepsilon)\Delta = \text{id}$

$$\varepsilon(\mathbf{2}) \otimes \mathbf{2} + \varepsilon(*) \otimes \mathbf{2} = \mathbf{2} \otimes \varepsilon(\mathbf{2}) + * \otimes \varepsilon(\mathbf{2}) = \mathbf{2}$$

If H is a Hopf algebra, then a H (right-) comodule is a vector space A with a map $\rho : A \rightarrow A \otimes H$ such that

$$(\rho \otimes \text{id})\rho = (\text{id} \otimes \Delta)\rho \quad \text{and} \quad (\text{id} \otimes \varepsilon)\rho = \text{id}.$$

Here Δ is a **coproduct** on H . ρ is a **coaction** on A .

MPLs **modulo** $i\pi$ form a Hopf algebra H . For the full space of MPLs, we need the comodule $\mathbb{Q}[i\pi] \otimes H$, with a coaction ρ where $\rho(i\pi) = i\pi \otimes 1$.

[Goncharov, Duhr, Brown]

In this presentation, the distinction is not terribly important. We continue to use Δ to denote our operations, which are formally coactions of bialgebras.

Hopf algebras

A **Hopf algebra** is a bialgebra H with an *antipode* map $S : H \rightarrow H$ that satisfies

$$\begin{array}{ccccc} & H \otimes H & \xrightarrow{S \otimes \text{id}} & H \otimes H & \\ & \uparrow \Delta & & \downarrow \mu & \\ H & \xrightarrow{\varepsilon} & K & \xrightarrow{\eta} & H \\ & \downarrow \Delta & & \uparrow \mu & \\ & H \otimes H & \xrightarrow{\text{id} \otimes S} & H \otimes H & \end{array}$$

(Here μ, η denote multiplication and inclusion, respectively.)

The incidence bialgebra becomes a Hopf algebra if we adjoin inverse elements $(S \subseteq S)^{-1}$.

Example of the incidence algebra: edges of graphs

Can also start with a cut diagram.

$$\Delta(12) = \mathbf{12} \otimes \mathbf{12} + \mathbf{1} \otimes \mathbf{12}$$

$$\Delta_{\text{Inc}} \left[\begin{array}{c} \text{---} \text{---} \\ \text{e}_1 \text{---} \text{---} \\ \text{---} \text{---} \\ \text{e}_2 \text{---} \text{---} \end{array} \right] = \begin{array}{c} \text{---} \text{---} \\ \text{e}_1 \text{---} \text{---} \\ \text{---} \text{---} \\ \text{e}_2 \text{---} \text{---} \end{array} \otimes \begin{array}{c} \text{---} \text{---} \\ \text{e}_1 \text{---} \text{---} \\ \text{---} \text{---} \\ \text{e}_2 \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} \text{---} \\ \text{e}_1 \text{---} \text{---} \\ \text{---} \text{---} \\ \text{e}_2 \text{---} \text{---} \end{array} \otimes \begin{array}{c} \text{---} \text{---} \\ \text{e}_1 \text{---} \text{---} \\ \text{---} \text{---} \\ \text{e}_2 \text{---} \text{---} \end{array}$$

$$\Delta(12) = \mathbf{12} \otimes \mathbf{12}$$

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Multiple polylogarithms (MPL)

A large class of iterated integrals are described by multiple polylogarithms:

$$G(a_1, \dots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_n; t)$$

Examples:

$$G(0; z) = \log z, \quad G(a; z) = \log \left(1 - \frac{z}{a} \right)$$

$$G(\vec{a}_n; z) = \frac{1}{n!} \log^n \left(1 - \frac{z}{a} \right), \quad G(\vec{0}_{n-1}, a; z) = -\text{Li}_n \left(\frac{z}{a} \right)$$

Harmonic polylog if all $a_i \in \{-1, 0, 1\}$.
 n is the *transcendental weight*.

Many Feynman integrals can be written in terms of classical and harmonic polylogs.

Closure under multiplication via the shuffle product:

$$G(\vec{a}_1; z) G(\vec{a}_2; z) = \sum_{\vec{a} \in \vec{a}_1 \amalg \vec{a}_2} G(\vec{a}; z),$$

where $\vec{a}_1 \amalg \vec{a}_2$ are the permutations preserving the relative orderings of \vec{a}_1 and \vec{a}_2 .

There is a coaction on MPLs. It is graded by weight, and thus breaks MPLs into simpler functions (lower weight).

$$\Delta(\log z) = 1 \otimes \log z + \log z \otimes 1$$

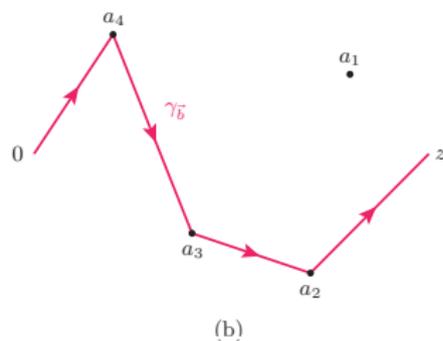
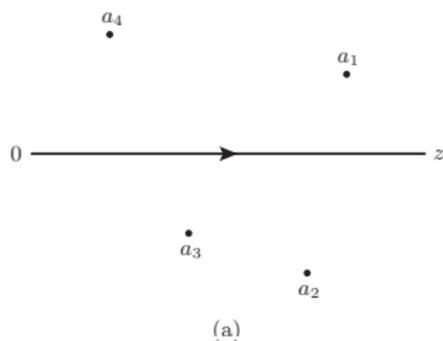
$$\Delta(\log x \log y) = 1 \otimes (\log x \log y) + \log x \otimes \log y + \log y \otimes \log x + (\log x \log y) \otimes 1$$

$$\Delta(\text{Li}_2(z)) = 1 \otimes \text{Li}_2(z) + \text{Li}_2(z) \otimes 1 + \text{Li}_1(z) \otimes \log z$$

Contour integrals

The coaction is a pairing of contours and integrands. Recalls the incidence algebra.

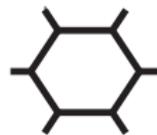
$$\Delta_{\text{MPL}}(G(\vec{a}; z)) = \sum_{\vec{b} \subseteq \vec{a}} G(\vec{b}; z) \otimes G_{\vec{b}}(\vec{a}; z)$$



Contour (b) takes a subset of residues in a given order.

A useful **basis** for all 1-loop integrals:

$$J_n = \frac{ie^{\gamma_E \epsilon}}{\pi^{D_n/2}} \int d^{D_n} k \prod_{j=1}^n \frac{1}{(k - q_j)^2 - m_j^2}$$



- k is the loop momentum
- q_j are sums of external momenta, m_j are internal masses
- Dimensions:

$$D_n = \begin{cases} n - 2\epsilon, & \text{for } n \text{ even,} \\ n + 1 - 2\epsilon, & \text{for } n \text{ odd.} \end{cases}$$

e.g. tadpoles and bubbles in $2 - 2\epsilon$ dimensions,
triangles and boxes in $4 - 2\epsilon$ dimensions, etc.

- Each J_n has uniform transcendental weight and satisfies nice differential equations.

2 equivalent Hopf algebras

The **combinatorial** algebra agrees with the Hopf algebra on the **MPL** of evaluated diagrams!

- The graph with n edges is interpreted as J_n , i.e. in D_n dimensions, no numerator.
- Need to insert extra terms in the diagrammatic equation:

$$\Delta \left[\text{Diagram with edges } e_1 \text{ and } e_2 \right] = \text{Diagram with edges } e_1 \text{ and } e_2 \otimes \text{Diagram with edges } e_1 \text{ and } e_2 \text{ with red cut lines} \\ + \text{Diagram with edge } e_1 \text{ in a circle} \otimes \left(\text{Diagram with edges } e_1 \text{ and } e_2 \text{ with red cut lines} + \frac{1}{2} \text{Diagram with edges } e_1 \text{ and } e_2 \text{ with red cut lines} \right) \\ + \text{Diagram with edge } e_2 \text{ in a circle} \otimes \left(\text{Diagram with edges } e_1 \text{ and } e_2 \text{ with red cut lines} + \frac{1}{2} \text{Diagram with edges } e_1 \text{ and } e_2 \text{ with red cut lines} \right)$$

Isomorphic to the more basic construction. (For any value of $1/2$.)

- How do we evaluate the cut graphs?

[related work: Brown; Bloch and Kreimer]

What are generalized cuts?

Traditional [Veltman]:

$$\begin{aligned} \bullet &= i & \circ &= -i \\ \bullet \xrightarrow{p} \bullet &= \frac{i}{p^2 + i\epsilon} & \circ \xrightarrow{p} \circ &= \frac{-i}{p^2 - i\epsilon} \\ \bullet \xrightarrow{p} \circ &= 2\pi \delta(p^2) \theta(p_0) \end{aligned}$$

Better interpretation: cuts should be understood primarily as **residues**.

[related: Kosower, Larsen]

- Change the **contour**, not the integrand
- Consistent with prior expectations
- Fits with diagrammatic coaction

Generalized cuts as residues and determinants

1-loop cuts defined as **residues**:

$$\mathcal{C}_C[I_n] = \frac{e^{\gamma_E \epsilon}}{i\pi^{\frac{D}{2}}} \int_{\Gamma_C} d^D k \prod_{j \notin C} \frac{1}{(k - q_j)^2 - m_j^2 + i0} \quad \text{mod } i\pi,$$

- C is the set of cut propagators
- Contour Γ_C encircles poles of cut propagators

Cut integrals give **discontinuities** of their uncut counterparts.

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Cut integrals give **discontinuities** of their uncut counterparts.

Some results:

$$\mathcal{C}_C I_n = \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_C}} \left(\frac{Y_C}{G_C} \right)^{(D-c)/2} \int \frac{d\Omega_{D-c+1}}{i\pi^{D/2}} \left[\prod_{j \notin C} \frac{1}{(k - q_j)^2 - m_j^2} \right]_C$$

where we often find Gram and modified Cayley determinants:

$$G_C = \det(q_i \cdot q_j)_{i,j \in C \setminus *}$$

$$Y_C = \det \left(\frac{1}{2}(m_i^2 + m_j^2 + (q_i - q_j)^2) \right)_{i,j \in C}$$

Maximal and next-to-maximal cuts

Some special cases:

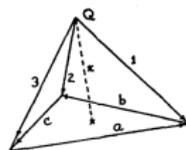
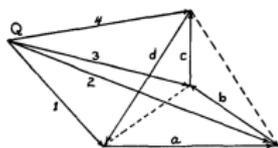
$$C_{2k}[J_{2k}] = \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_{2k}}} \left(\frac{Y_{2k}}{G_{2k}} \right)^{-\epsilon}$$

$$C_{2k+1}[J_{2k+1}] = \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon) \sqrt{G_{2k+1}}} \left(\frac{Y_{2k+1}}{G_{2k+1}} \right)^{-\epsilon}.$$

$$C_{2k-1}[J_{2k}] = -\frac{\Gamma(1-2\epsilon)^2}{\Gamma(1-\epsilon)^3} \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_{2k}}} \left(\frac{Y_{2k-1}}{G_{2k-1}} \right)^{-\epsilon} {}_2F_1 \left(\frac{1}{2}, -\epsilon; 1-\epsilon; \frac{G_{2k} Y_{2k-1}}{Y_{2k} G_{2k-1}} \right).$$

For more complicated cuts, we set up a Feynman parametrization.

Landau conditions are expressed in [polytope geometry](#): these determinants are volumes of simplices. [\[Cutkosky\]](#)



$$\alpha_i \left[(k^E - q_i^E)^2 + m_i^2 \right] = 0, \quad \forall i.$$

$$\sum_{i=1}^n \alpha_i (k^E - q_i^E) = 0.$$

Therefore

$$\begin{pmatrix} (k^E - q_1^E) \cdot (k^E - q_1^E) & \dots & (k^E - q_1^E) \cdot (k^E - q_c^E) \\ \vdots & \ddots & \vdots \\ (k^E - q_c^E) \cdot (k^E - q_1^E) & \dots & (k^E - q_c^E) \cdot (k^E - q_c^E) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_c \end{pmatrix} = 0.$$

Nontrivial solution $\Leftrightarrow Y_C = 0$ and integral over Γ_C : Landau singularities of the first type.

Second-type singularities come from $G_C = 0$ and integral over $\Gamma_{C \cup \infty}$. Contour is pinched at infinity.

Homology theory for Feynman contours

Homology describes the inequivalent integration contours. Also explains why cuts are discontinuities. Use **Leray residues**. [Fotiadi, Pham; Hwa, Teplitz; Federbusch; Eden, Fairlie, Landshoff, Nuttall, Olive, Polkinghorne,...]

Residues: if

$$\omega = \frac{ds}{s} \wedge \psi + \theta$$

and $S = \{s = 0\}$, while ψ, θ are regular on S , then

$$\text{Res}_S[\omega] = \psi|_S.$$

Cut integrals: if $I_n = \int \omega_n$, then

$$\begin{aligned} \mathcal{C}_C[I_n] &= \int_{S_C} \text{Res}_{S_C}[\omega_n] \\ &= (2\pi i)^{-k} \int_{\delta S_C} \omega_n \end{aligned}$$

where δ constructs a “tubular neighborhood” around $S_C = \bigcap_{i \in C} S_i$, the spherical locus of the cut conditions.

For 1-loop Feynman integrals, the [Decomposition Theorem](#) shows that the contours $\Gamma_C = \delta S_C$ form a basis. [\[Fotiadi, Pham\]](#)

$$\Gamma_{\infty C} = -2x_C \Gamma_C - \sum_{C \subset X \subseteq [n]} (-1)^{\lceil \frac{|C|}{2} \rceil + \lceil \frac{|X|}{2} \rceil} \Gamma_X, \quad x_C = \begin{cases} 1, & |C| \text{ odd,} \\ 0, & |C| \text{ even.} \end{cases}$$

Can work in a compactified projective space, where ∞ is on the same footing as other labels.

Examples of the graphical conjecture

$$\begin{aligned}
 \Delta \left[\text{Diagram} \right] &= \text{Diagram} \otimes \text{Diagram} \\
 &+ \text{Diagram} \otimes \left(\text{Diagram} + \frac{1}{2} \text{Diagram} \right) \\
 &+ \text{Diagram} \otimes \left(\text{Diagram} + \frac{1}{2} \text{Diagram} \right)
 \end{aligned}$$

The diagrams are:

- A lens-shaped diagram with two vertices and two edges labeled e_1 and e_2 .
- A circle with a vertical line extending downwards, labeled e_1 or e_2 .
- A lens-shaped diagram with two vertices and two edges labeled e_1 and e_2 , with red vertical lines on each edge.

$$\begin{aligned}
 \Delta \left(\int_{\Gamma_\emptyset} \omega_{12} \right) &= \int_{\Gamma_\emptyset} \omega_{12} \otimes \int_{\Gamma_{12}} \omega_{12} + \int_{\Gamma_\emptyset} \omega_1 \otimes \left(\int_{\Gamma_1} \omega_{12} + \frac{1}{2} \int_{\Gamma_{12}} \omega_{12} \right) + \dots \\
 &= \int_{\Gamma_\emptyset} \omega_{12} \otimes \int_{\Gamma_{12}} \omega_{12} + \int_{\Gamma_\emptyset} \omega_1 \otimes \int_{-\frac{1}{2}\Gamma_{1\infty}} \omega_{12} + \int_{\Gamma_\emptyset} \omega_2 \otimes \int_{-\frac{1}{2}\Gamma_{2\infty}} \omega_{12}
 \end{aligned}$$

The odd-shaped integrals have poles at infinity.

Examples of the graphical conjecture

$$\Delta \left[\text{Diagram} \right] = \text{Diagram}_1 \otimes \text{Diagram}_2 + \text{Diagram}_3 \otimes \text{Diagram}_4 + \text{Diagram}_5 \otimes \text{Diagram}_6 + \text{Diagram}_7 \otimes \text{Diagram}_8$$

The diagram on the left is a triangle with vertices labeled 1, 2, and 3. The edges are labeled e_1 , e_2 , and e_3 . The edges e_2 and e_3 are thick lines, while e_1 is a thin line. The triangle is enclosed in large square brackets.

The right-hand side consists of eight terms, each being a tensor product (\otimes) of two diagrams:

- Term 1: A bubble diagram with vertices 1 and 1, edges e_2 (top) and e_1 (bottom), and a thick line on the right.
- Term 2: A triangle with vertices 1, 2, and 3. Edges e_2 and e_3 are thick lines. Edge e_1 is a thin line. Dashed red lines connect the vertices to the edges e_2 and e_3 .
- Term 3: A bubble diagram with vertices 2 and 2, edges e_3 (top) and e_2 (bottom), and a thick line on the right.
- Term 4: A triangle with vertices 1, 2, and 3. Edges e_2 and e_3 are thick lines. Edge e_1 is a thin line. Dashed red lines connect the vertices to the edges e_2 and e_3 .
- Term 5: A bubble diagram with vertices 3 and 3, edges e_3 (top) and e_1 (bottom), and a thick line on the right.
- Term 6: A triangle with vertices 1, 2, and 3. Edges e_2 and e_3 are thick lines. Edge e_1 is a thin line. Dashed red lines connect the vertices to the edges e_2 and e_3 .
- Term 7: A triangle with vertices 1, 2, and 3. Edges e_2 and e_3 are thick lines. Edge e_1 is a thin line. Dashed red lines connect the vertices to the edges e_2 and e_3 .
- Term 8: A triangle with vertices 1, 2, and 3. Edges e_2 and e_3 are thick lines. Edge e_1 is a thin line. Dashed red lines connect the vertices to the edges e_2 and e_3 .

Terms with 1/2 are always present in principle, but vanished here due to massless propagators.

Examples of the graphical conjecture

$$\Delta \left[\begin{array}{c} \text{1} \\ \text{e}_1 \quad \text{e}_2 \\ \text{e}_3 \end{array} \right] = \begin{array}{c} \text{1} \\ \text{e}_1 \quad \text{e}_2 \\ \text{e}_3 \end{array} \otimes \begin{array}{c} \text{1} \\ \text{e}_1 \quad \text{e}_2 \\ \text{e}_3 \end{array} .$$

Statement of the graphical conjecture

The coaction on 1-loop graphs defined by pinching and cutting subsets of propagators,

when evaluated by Feynman rules,
if expanded order by order in ϵ ,

is consistent with the coaction on MPLs!

Evidence for the graphical conjecture

- all tadpoles and bubbles
- triangles and boxes with several combinations of internal and external masses
- consistency checks for more complicated boxes, 0m pentagon, 0m hexagon
- diagrammatic groupings emerging in 2-loop integrals

Checked to several orders in ϵ , or for closed forms with hypergeometric functions.

Coproducts of diagrams

$$\Delta \left[\begin{array}{c} \text{---} 1 \text{---} \\ \text{---} e_2 \text{---} \diagup \text{---} 2 \text{---} \\ \text{---} e_3 \text{---} \text{---} \\ \text{---} e_1 \text{---} \diagdown \text{---} 3 \text{---} \end{array} \right] = \begin{array}{c} \text{---} 1 \text{---} \text{---} e_2 \text{---} \text{---} 1 \text{---} \\ \text{---} e_1 \text{---} \end{array} \otimes \begin{array}{c} \text{---} 1 \text{---} \text{---} e_2 \text{---} \text{---} 2 \text{---} \\ \text{---} e_1 \text{---} \text{---} e_3 \text{---} \text{---} 3 \text{---} \end{array} + \begin{array}{c} \text{---} 2 \text{---} \text{---} e_3 \text{---} \text{---} 2 \text{---} \\ \text{---} e_2 \text{---} \end{array} \otimes \begin{array}{c} \text{---} 1 \text{---} \text{---} e_2 \text{---} \text{---} 2 \text{---} \\ \text{---} e_1 \text{---} \text{---} e_3 \text{---} \text{---} 3 \text{---} \end{array} \\ + \begin{array}{c} \text{---} 3 \text{---} \text{---} e_3 \text{---} \text{---} 3 \text{---} \\ \text{---} e_1 \text{---} \end{array} \otimes \begin{array}{c} \text{---} 1 \text{---} \text{---} e_2 \text{---} \text{---} 2 \text{---} \\ \text{---} e_1 \text{---} \text{---} e_3 \text{---} \text{---} 3 \text{---} \end{array} + \begin{array}{c} \text{---} 1 \text{---} \text{---} e_2 \text{---} \text{---} 2 \text{---} \\ \text{---} e_1 \text{---} \text{---} e_3 \text{---} \text{---} 3 \text{---} \end{array} \otimes \begin{array}{c} \text{---} 1 \text{---} \text{---} e_2 \text{---} \text{---} 2 \text{---} \\ \text{---} e_1 \text{---} \text{---} e_3 \text{---} \text{---} 3 \text{---} \end{array}$$

Second entries **are** discontinuities; first entries **have** discontinuities.

Coproducts of diagrams

$$\Delta \left[\begin{array}{c} \text{---} 1 \text{---} \\ \begin{array}{l} e_2 \nearrow \\ e_1 \searrow \end{array} \\ \begin{array}{l} \text{---} 2 \text{---} \\ \text{---} 3 \text{---} \end{array} \\ e_3 \end{array} \right] = \begin{array}{c} \text{---} 1 \text{---} \\ \begin{array}{l} e_2 \nearrow \\ e_1 \searrow \end{array} \\ \text{---} 1 \text{---} \end{array} \otimes \begin{array}{c} \text{---} 1 \text{---} \\ \begin{array}{l} e_2 \nearrow \\ e_1 \searrow \end{array} \\ \begin{array}{l} \text{---} 2 \text{---} \\ \text{---} 3 \text{---} \end{array} \\ e_3 \end{array} + \begin{array}{c} \text{---} 2 \text{---} \\ \begin{array}{l} e_3 \nearrow \\ e_2 \searrow \end{array} \\ \text{---} 2 \text{---} \end{array} \otimes \begin{array}{c} \text{---} 1 \text{---} \\ \begin{array}{l} e_2 \nearrow \\ e_1 \searrow \end{array} \\ \begin{array}{l} \text{---} 2 \text{---} \\ \text{---} 3 \text{---} \end{array} \\ e_3 \end{array} \\ + \begin{array}{c} \text{---} 3 \text{---} \\ \begin{array}{l} e_3 \nearrow \\ e_1 \searrow \end{array} \\ \text{---} 3 \text{---} \end{array} \otimes \begin{array}{c} \text{---} 1 \text{---} \\ \begin{array}{l} e_2 \nearrow \\ e_1 \searrow \end{array} \\ \begin{array}{l} \text{---} 2 \text{---} \\ \text{---} 3 \text{---} \end{array} \\ e_3 \end{array} + \begin{array}{c} \text{---} 1 \text{---} \\ \begin{array}{l} e_2 \nearrow \\ e_1 \searrow \end{array} \\ \begin{array}{l} \text{---} 2 \text{---} \\ \text{---} 3 \text{---} \end{array} \\ e_3 \end{array} \otimes \begin{array}{c} \text{---} 1 \text{---} \\ \begin{array}{l} e_2 \nearrow \\ e_1 \searrow \end{array} \\ \begin{array}{l} \text{---} 2 \text{---} \\ \text{---} 3 \text{---} \end{array} \\ e_3 \end{array}$$

Second entries **are** discontinuities; first entries **have** discontinuities.

Motivated by the identity

$$\Delta \text{ Disc} = (\text{Disc} \otimes 1) \Delta.$$

The companion relation

$$\Delta \partial = (1 \otimes \partial) \Delta$$

produces differential equations.

$$\Delta \text{ Disc} = (\text{Disc} \otimes 1) \Delta$$

$$\Delta (\text{Disc } I_n) = (\text{Disc} \otimes 1) (\Delta I_n)$$

Since $\Delta (\text{Disc } I_n) = 1 \otimes (\text{Disc } I_n) + \dots$, it is enough to look at the terms $\Delta_{1,w-1} I_n$.

The basis integrals of weight 1 are precisely the tadpoles and bubbles. The corresponding cut diagrams have 1 or 2 propagators cut.

Therefore: the discontinuities are precisely the unitarity cut diagrams (momentum invariant discontinuities) and the single-cut diagrams (mass discontinuities).

Generalized cuts can be interpreted as well.

Application: differential equations

$$\Delta \partial = (1 \otimes \partial) \Delta$$

Likewise, we get differential equations by focusing on nearly-maximal cuts in the second factor:

$$\begin{aligned} d \left[\text{pentagon} \right] &= \sum_{(ijk)} \left[\text{triangle} \right]_i^j \left[\text{hexagon} \right]_{j,k}^i + \frac{1}{2} \sum_l \left[\text{hexagon} \right]_{j,k}^i \left[\text{hexagon} \right]_{j,k}^i \right]_{\epsilon^0} \\ &+ \sum_{(ijkl)} \left[\text{rectangle} \right]_{i,k}^j \left[\text{hexagon} \right]_{j,k}^i \left[\text{hexagon} \right]_{j,k}^i \right]_{\epsilon^0} + \epsilon \left[\text{pentagon} \right] \left[\text{hexagon} \right]_{j,k}^i \left[\text{hexagon} \right]_{j,k}^i \right]_{\epsilon^1} \end{aligned}$$

This also shows a way to identify the symbol alphabet.

The master formula for the ${}_2F_1$ family

Consider the diagrammatic coaction

$$\Delta \left[\begin{array}{c} e_2 \\ \text{---} 1 \text{---} \\ e_1 \end{array} \middle| e_3 \right] = \begin{array}{c} \circlearrowleft e_1 \\ | \end{array} \otimes \left(\begin{array}{c} e_2 \\ \text{---} 1 \text{---} \\ e_1 \end{array} \middle| e_3 + \frac{1}{2} \begin{array}{c} e_2 \\ \text{---} 1 \text{---} \\ e_1 \end{array} \middle| e_3 \right) \\ + \begin{array}{c} e_1 \\ \text{---} 1 \text{---} \\ e_2 \end{array} \otimes \begin{array}{c} e_2 \\ \text{---} 1 \text{---} \\ e_1 \end{array} \middle| e_3$$

There is a coaction on ${}_2F_1$ that gives

$$\begin{aligned} \Delta {}_2F_1(1, 1 + \epsilon, 2 - \epsilon, x) &= {}_2F_1(1, \epsilon, 1 - \epsilon, x) \otimes {}_2F_1(1, 1 + \epsilon, 2 - \epsilon, x) \\ &\quad + {}_2F_1(1, 1 + \epsilon, 2 - \epsilon, x) \otimes {}_2F_1\left(1, \epsilon, 1 - \epsilon, \frac{1}{x}\right) \end{aligned}$$

without expanding in ϵ !

Coaction of the form

$$\Delta\left(\int_{\gamma} \omega\right) = \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \omega$$

with a duality condition

$$P_{ss} \int_{\gamma_i} \omega_j = \delta_{ij}.$$

P_{ss} is semi-simple projection (“drop logarithms but not π ”).

To be precise, P_{ss} projects onto the space of semi-simple numbers x satisfying $\Delta(x) = x \otimes 1$.

The master formula for the ${}_2F_1$ family

Consider the family of integrands

$$\omega(\alpha_1, \alpha_2, \alpha_3) = x^{\alpha_1}(1-x)^{\alpha_2}(1-zx)^{\alpha_3} dx$$

where $\alpha_j = n_j + \epsilon_j$ and $n_j \in \mathbb{Z}$.

$$\int_0^1 \omega(\alpha_1, \alpha_2, \alpha_3) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2 - \alpha_1)}{\Gamma(\alpha_2)} {}_2F_1(-\alpha_3, \alpha_1 + 1; \alpha_2 + \alpha_1 + 2; z)$$

Basis of master integrands:

$$\int_0^1 \omega = c_0 \int_0^1 \omega_0 + c_1 \int_0^1 \omega_1$$

where

$$\begin{aligned}\omega_0 &= x^{\epsilon_1}(1-x)^{-1+\epsilon_2}(1-zx)^{\epsilon_3} \\ \omega_1 &= x^{\epsilon_1}(1-x)^{\epsilon_2}(1-zx)^{-1+\epsilon_3}\end{aligned}$$

With the two contours $\gamma_0 = [0, 1]$ and $\gamma_1 = [0, 1/z]$, we have $P_{ss} \int_{\gamma_i} \omega_j \sim \delta_{ij}$.

Master formula for Appell F_1

Family of integrands for F_1 .

$$\omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = x^{\alpha_1}(1-x)^{\alpha_2}(1-z_1x)^{\alpha_3}(1-z_2x)^{\alpha_4} dx$$

where $\alpha_j = n_j + \epsilon_j$ and $n_j \in \mathbb{Z}$.

$$\int_0^1 \omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2 - \alpha_1)}{\Gamma(\alpha_2)} F_1(\alpha_1, \alpha_3, \alpha_4, \alpha_2; z_1, z_2)$$

Master integrands:

$$\omega_0 = x^{\epsilon_1}(1-x)^{-1+\epsilon_2}(1-z_1x)^{\epsilon_3}(1-z_2x)^{\epsilon_4}$$

$$\omega_1 = x^{\epsilon_1}(1-x)^{\epsilon_2}(1-z_1x)^{-1+\epsilon_3}(1-z_2x)^{\epsilon_4}$$

$$\omega_2 = x^{\epsilon_1}(1-x)^{\epsilon_2}(1-z_1x)^{\epsilon_3}(1-z_2x)^{-1+\epsilon_4}$$

Master contours: $\gamma_0 = [0, 1]$, $\gamma_1 = [0, z_1^{-1}]$, $\gamma_2 = [0, z_2^{-1}]$.

Diagrammatic example with F_1

$$\begin{aligned}
 \Delta \left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right] &= \textcircled{e_1} \otimes \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} + \frac{1}{2} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \\
 &+ \textcircled{e_2} \otimes \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} + \frac{1}{2} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \\
 &+ \begin{array}{c} \text{---} \\ \text{---} \end{array} \otimes \begin{array}{c} \text{---} \\ \text{---} \end{array} \\
 &+ \begin{array}{c} \text{---} \\ \text{---} \end{array} \otimes \begin{array}{c} \text{---} \\ \text{---} \end{array}
 \end{aligned}$$

The diagrammatic equation shows the expansion of a triangle with edges e_1 , e_2 , and e_3 . The left side is a triangle with a thick line on the left edge. The right side is a sum of four terms:

- Term 1: A circle labeled e_1 with a vertical line extending downwards, tensored with a sum of two triangles. The first triangle has a thick line on the left edge and a dashed line on the bottom edge. The second triangle has a thick line on the left edge and dashed lines on both the bottom and top edges.
- Term 2: A circle labeled e_2 with a vertical line extending downwards, tensored with a sum of two triangles. The first triangle has a thick line on the left edge and a dashed line on the top edge. The second triangle has a thick line on the left edge and dashed lines on both the top and bottom edges.
- Term 3: A diagram consisting of a horizontal line with a loop (a circle) attached to it, labeled e_1 at the top and e_2 at the bottom, tensored with a triangle with a thick line on the left edge and dashed lines on the bottom and top edges.
- Term 4: A triangle with a thick line on the left edge, tensored with a triangle with a thick line on the left edge and dashed lines on the bottom and top edges.

Master formula for ${}_p+1F_p$

Family of integrands for ${}_3F_2$.

$$\omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = x^{\alpha_1}(1-x)^{\alpha_2}y^{\alpha_3}(1-y)^{\alpha_4}(1-zxy)^{\alpha_5} dx dy$$

where $\alpha_i = n_i + \epsilon_i$ and $n_i \in \mathbb{Z}$.

Then

$$\int_0^1 \int_0^1 \omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \frac{\Gamma(\epsilon_1)\Gamma(\epsilon_2)\Gamma(\epsilon_3)\Gamma(\epsilon_4)}{\Gamma(\epsilon_1)\Gamma(\epsilon_2)} {}_3F_2(\alpha_1 + 1, \alpha_3 + 1, -\alpha_5; 2 + \alpha_1 + \alpha_2, 2 + \alpha_3 + \alpha_4; z)$$

Basis of master integrands:

$$\omega_0 = x^{\epsilon_1}(1-x)^{-1+\epsilon_2}y^{\epsilon_3}(1-y)^{-1+\epsilon_4}(1-zxy)^{\epsilon_5}$$

$$\omega_1 = x^{\epsilon_1}(1-x)^{-1+\epsilon_2}y^{\epsilon_3}(1-y)^{\epsilon_4}(1-zxy)^{-1+\epsilon_5}$$

$$\omega_2 = x^{\epsilon_1}(1-x)^{\epsilon_2}y^{\epsilon_3}(1-y)^{-1+\epsilon_4}(1-zxy)^{-1+\epsilon_5}$$

With the master contours $\gamma_0 = \int_0^1 dx \int_0^1 dy$, $\gamma_1 = \int_0^1 dx \int_0^{1/zx} dy$,

$\gamma_2 = \int_0^1 dy \int_0^{1/zy} dx$, we find that $P_{ss} \int_{\gamma_i} \omega_j \sim \delta_{ij}$

Diagrammatic example with ${}_3F_2$

$$\Delta \left[\begin{array}{c} \text{---} \\ | \\ \text{---} \\ / \quad \backslash \\ \text{---} \\ \backslash \quad / \\ \text{---} \\ | \\ \text{---} \\ \text{---} \end{array} \right] =$$

(with various prefactors and dimension shifts inserted to produce pure integrals)

Features of diagrammatic coaction at two loops

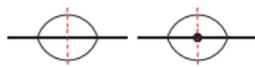
Matrix of integrands and contours for *each topology*.

Example: sunrise with one internal mass. 2 master integrands in top topology.

$$\begin{aligned} \text{---} \bigcirc \text{---} &= \int_{\Gamma_\emptyset} \omega_{111} \sim {}_2F_1 \left(1 + 2\epsilon, 1 + \epsilon, 1 - \epsilon, p^2/m^2 \right) \\ \text{---} \bullet \bigcirc \text{---} &= \int_{\Gamma_\emptyset} \omega_{121} \sim {}_2F_1 \left(2 + 2\epsilon, 1 + \epsilon, 1 - \epsilon, p^2/m^2 \right) \end{aligned}$$

For each, only two of the generalized cuts are linearly independent!

Thus 2 independent integration contours, e.g. Γ_\emptyset and Γ_{123} .



Diagonalize the matrix: $\int_{\gamma_i} \omega_j \sim \delta_{ij}$ with

$$\begin{aligned} \omega_1 &= a\epsilon^2 \omega_{111}, & \omega_2 &= b\epsilon \omega_{111} + c\epsilon \omega_{121} \\ \gamma_1 &= \Gamma_\emptyset, & \gamma_2 &= -\frac{1}{6\epsilon} \Gamma_{123} + \frac{2}{3} \Gamma_\emptyset \end{aligned}$$

Coaction $\Delta \left(\int_\gamma \omega \right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$ is expressible in terms of **diagrams**.

Features of diagrammatic coaction at two loops

For example:

$$\begin{aligned}
 \Delta \left(\text{Diagram} \right) = & \text{Diagram} \otimes \left[\text{Diagram} + \text{Diagram} \right] \\
 & + \left[\text{Diagram} + \text{Diagram} \right] \otimes \left[\text{Diagram} \right] \\
 & + \left[\text{Diagram} + \text{Diagram} + \text{Diagram} \right]
 \end{aligned}$$

The diagrams are two-loop self-energy diagrams. The first row shows the original diagram and its tensor product with a sum of two diagrams: one with a dot on the horizontal line and one with a dot on the vertical line. The second row shows the tensor product of a sum of two diagrams (one with a dot on the horizontal line, one with a dot on the vertical line) with a diagram that has a vertical dashed red line. The third row shows a sum of three diagrams: one with a dot on the horizontal line, one with a vertical dashed red line, and one with a dot on the horizontal line and a vertical dashed red line.

(with prefactors as seen on previous slide)

In particular, we can recover weight 1 discontinuities:

$$\Delta_{1,k-1} \left(\text{Diagram} \right) = \log(p^2 - m^2) \otimes \text{Diagram} + \log(m^2) \otimes \text{Diagram}$$

The diagrams in the equation are two-loop self-energy diagrams with a vertical dashed red line. The first diagram has the dashed line on the left vertical line, and the second has it on the right vertical line.

Summary & Outlook

- We observe a **Hopf algebra structure on Feynman diagrams**. At 1 loop, there is a basis for which the coaction is simply related to **pinches and cuts** of the original diagram. Beyond 1-loop: encounter matrix equations (cf. higher-order differential equations)
- Corresponds to Goncharov's Hopf algebra on MPLs, with prospects for extensions to hypergeometric integrals and beyond.
- Cuts should be understood through homology and Leray residues.
- Deep connections to discontinuities and differential equations, which are tools for computation.
- Abstracted master formula: a Hopf algebra based on matched **pairs of integrands and contours**
- To explore further: systematic description beyond 1 loop, full range of hypergeometric functions, applications to integral and amplitude computations.