1 Questions exercise class (29.9.2019)

1.1 Meaning of $\frac{\partial x'^{\mu}}{\partial x^{\nu}}$

We have seen that $\frac{\partial x_\mu}{\partial x_\nu} = \delta^\nu_\mu$. *However from classical physics I know that* $\frac{\partial x_1}{\partial x_0}$ *is the velocity along the direction* 1. *From these two it would follow that the velocity along* 1 *is always zero?*

In GR a different approach is taken. The trajectory of a massive point object is coded in a curve $\gamma(\tau)$ along the four-dimensional (curved) spacetime. Thus the velocity of that object is denoted by

$$v = \frac{d}{d\tau}\gamma(\tau) \tag{1}$$

Instead, $\frac{\partial x'^{\mu}}{\partial x^{\nu}}$ should be interpreted as a component of the jacobian of a transformation $x' = x'(x^0, x^1, x^2, x^3)$ of the coordinates.

1.2 Meaning of $(\partial_\mu \omega_\rho)'$

Why does $(\partial_\mu \omega_\rho)' = \partial'_\mu \omega'_\rho$*?*

By definition. One of the axioms of GR is that its equations should be covariant, that is they should involve tensors, that is objects which transform under the rule

$$\omega'_{\mu_1 \dots \mu_n} = \omega_{\sigma_1 \dots \sigma_n} \prod_{i=1}^{n} \frac{\partial x_{\mu_i}}{\partial x'_{\sigma_i}} \qquad v'^{\mu_1 \dots \mu_n} = v^{\sigma_1 \dots \sigma_n} \prod_{i=1}^{n} \frac{\partial x'^{\mu_i}}{\partial x^{\sigma_i}}; \tag{2}$$

in the case of a two rank tensor

$$\omega'_{\mu\nu} = \omega_{\sigma\rho} \frac{\partial x_\mu}{\partial x'_\sigma} \frac{\partial x_\nu}{\partial x'_\rho}. \tag{3}$$

This request allows to cast equations for tensors in a form that is explicitely invariant under action of diffeomorfisms (trasnformations of coordinates). Indeed, the contraction of two tensors, $C^{\mu_1 \cdots \mu_n} H_{\mu_1, \dots, \mu_n}$ is invariant under transformations in the form (2). In trying to find a way to differentiate tensors while preserving covariance, we would like to find tensors whose components transform as given in (3). In the exercise classes we have shown that $\partial_\mu \omega_\nu$ does not have the transformation properties of a two rank tensor. Instead, we showed that this can be done by defining D_μ with the constraint

$$(D_\mu v_\nu)' := D'_\mu v'_\nu \stackrel{!}{=} D_\sigma v_\rho \frac{\partial x_\mu}{\partial x'_\sigma} \frac{\partial x_\nu}{\partial x'_\rho}. \tag{4}$$