Particle Physics in the Early Universe (FS 17)

Particles, groups and relativistic wave equations

Dr. Achilleas Lazopoulos

December 20, 2019

According to Wigner's work on the unitary representations of the Poincaré group, representations of it are characterized by a minimal set of Casimir invariants, $P_\mu P^\mu$ and $W_\mu W^\mu$, corresponding to mass and spin. Together with the notion that particles are representations of the symmetry group of the theory (and leaving gauge symmetry aside), one understands that particles are defined uniquely by their mass and spin.

Exercise 1

- Let $\hat{P}_\mu$ be the generator of translations along the direction μ, that is

$$e^{iaP_\mu}\Phi(x_0, x_1, x_2, x_3) = \Phi(x_0 + \delta_{\mu 0}a,\ x_1 + \delta_{\mu 1}a,\ x_2 + \delta_{\mu 2}a,\ x_3 + \delta_{\mu 3}a). \tag{1}$$

  Show that the eigenvalue equation for the Casimir invariant $P_\mu P^\mu \Phi = m^2\Phi$ is the Klein-Gordon equation.

- Let $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}J^{\nu\sigma}P^\rho$, where

$$J^{\nu\sigma} = (X \wedge P)_{\nu\sigma} = \epsilon_{\nu\sigma\tau\alpha}X^\tau P^\alpha \tag{2}$$

  is the relativistic angular momentum. Show that, if the particle Ψ is massive, in its rest frame

$$W_0\Phi = 0, \quad W_i\Phi = \frac{m}{2}\epsilon_{ijk}J^{jk}\Phi \tag{3}$$

  where J^{jk} is now the ordinary 3d Angular momentum. As a consequence show that the operator $W_\mu W^\mu$ reduces to the usual Casimir invariant for the angular momentum

$$W_\mu W^\mu = -m^2 J^{jk}J_{jk}. \tag{4}$$

  The eigenvalues of this operator denote the representation of the $SO(3)$ group, and thus $W_\mu W^\mu = s(s+1)$. If the particle is massless, a more careful treatment has to be done.

Exercise 2 In the last exercise, we have seen that imposing symmetry under Poincaré transformations naturally leads to viewing particles as representations of the Poincaré group. However, we have also seen that the spin of such particles can only be an integer. We would now like to construct half-spin particles. These particles must still satisfy the relation $P_\mu P^\mu \Psi = m^2 \Psi$.

- A topological space is said to be simply connected if every closed curve on it can be continuously contracted to a point. We will identify the group $SO(3)$ with a topological space, and show that it is not simply connected. Consider a solid sphere of radius π (meaning all the points on the surface and those inside the surface). We can associate to each rotation, given by a unit vector $\hat{n}$ and an angle $\theta \in [-\pi, \pi]$, the point in the solid sphere $\theta \hat{n}$. However, we observe that, for fixed $\hat{n}$, $\theta = \pi$ and $\theta = -\pi$ identify the exact same rotations, and thus one has that all the antipodal points on the sphere surface must be identified. The solid sphere, with this identification of antipodal points, is a topological space homeomorphic to the $SO(3)$ group. Show that it is not simply connected.

As a consequence of the fact that $SO(3)$ is not simply connected one obtains that $SO_+(3,1)$ (the ortochronous Lorentz group), which has as a subgroup $SO(3)$, is not simply connected. The universal covering space of $SO_+(3,1)$ is $SL(2,C)$.

- Show that the Lie algebra of $SO_+(3,1)$ is $su(2) \times su(2)$.

The relevance of covering spaces is especially clear in light of this result, that $SL(2,C)$ and $SO_+(3,1)$ share the same Lie algebra, $su(2) \times su(2)$. Indeed, representations of $su(2) \times su(2)$ exponentiate to representations of $SO_+(3,1)$ only if the sum of the spins j_1 and j_2 of the two $su(2)$ algebras satisfy $j_1 + j_2 \in \mathbb{Z}$, due to it being not simply connected, while they always exponentiate to representations of $SL(2,C)$.

- Consider the most general first order equation

$$A^\mu \partial_\mu \Psi - m\Psi = 0 \tag{5}$$

and show that this equation is invariant under transformations $\Psi \to S\Psi$, where S is a matrix in some representation of SL(2,C), if and only if

$$S^{-1}A^{\mu}S = \Lambda_{\mu}{}^{\nu}A_{\nu}. \tag{6}$$

It turns out that the above equation can only be satisfied if S is in a half-spin representation of SL(2,C). This relation defines a set of matrices A^{μ}, which give rise to a Clifford algebra.