Particle Physics in the Early Universe (FS 17)

Exercise Series 8

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In the following exercise we will lay the groundwork for deriving Majorana solutions to the Dirac equation, comparing them with general solutions to it. The difference between the two solutions lays entirely in the Majorana condition, which constrains the form of the general Dirac bi-spinor, and leads to many interesting properties that might be useful to describe specific SM particles.

Exercise 1 In this exercise we will show what are the properties of Majorana bi-spinors and why they are suited to describing left-handed massive neutrinos.

- Consider the gamma matrices in the Weyl representation

$$\gamma_\mu = \begin{bmatrix} 0 & \sigma_\mu \\ \sigma_\mu & 0 \end{bmatrix}, \tag{1}$$

  and show that in this representation the Dirac Lagrangian can be written as

$$\bar{\Psi} i \gamma^\mu \partial_\mu \Psi - m \bar{\Psi}\Psi = i\eta_1^\dagger \sigma^\mu \partial_\mu \eta_1 + i\eta_2^\dagger \sigma^\mu \partial_\mu \eta_2 - m(\eta_1^\dagger \eta_2 + \eta_2^\dagger \eta_1) \tag{2}$$

- Consider the gamma matrices in the Majorana representation

$$\gamma_0 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix} \quad \gamma_1 = \begin{bmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{bmatrix} \quad \gamma_2 = \begin{bmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{bmatrix} \quad \gamma_3 = \begin{bmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{bmatrix} \tag{3}$$

  Show that the Dirac equation with gamma matrices in this representation admits real solutions, $\Psi = \Psi^\star$ and conclude from this whether Ψ, satisfying this specific condition, has anti-particles or not. This condition is called the Majorana condition.

- Two representations of the gamma matrices are associated by a unitary matrix $\gamma'^\mu = U^\dagger \gamma^\mu U$. Show that the solution Ψ of Dirac's equation in the Majorana representation becomes then $U^\dagger\Psi$ in the new representation. Furthermore, show that the Majorana condition $\Psi = \Psi^\star$ in the new representation becomes $\Psi = UU^T\Psi^\star$. One can rewrite $UU^T = C\gamma^0$, which defines C: can you explain why C defines the operation of charge conjugation?

- One can show that if we transform from the Majorana to the Weyl representation,

$$UU^T = \begin{bmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{bmatrix}. \tag{4}$$

  Use the Majorana condition in this representation to constrain the form of the spinor $\Psi^T = (\vec{\eta}_1, \vec{\eta}_2)$ and derive a condition between $\vec{\eta}_1$ and $\vec{\eta}_2$. Use this new constrained condition to show that the Dirac Lagrangian, with gamma matrices in the Weyl representation, can be rewritten as

$$\bar{\Psi} i\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi = i\eta_1^\dagger \sigma^\mu \partial_\mu \eta_1 - im\eta_1^T \sigma^2 \eta_1 + h.c. \tag{5}$$

  using that $\omega^T A^\mu \partial_\mu \omega^\star = -\partial_\mu \omega^\dagger A^{T\mu} \omega$ if ω is a fermion field (can you show this?) and that $(\sigma^2 \sigma^\mu \sigma^2)^T = \sigma^\mu$. How is the mass term of this free lagrangian different from the one of the Dirac Lagrangian were no Majorana condition has been imposed?