Particle Physics in the Early Universe (FS 17)

Exercise Series 7

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Exercise 1 In this exercise we will show that in a Grand Unified Theory with a single coupling, that has the SM group as a subgroup, there is a condition between the coupling constants stemming from pure group theory normalization. Let us define the set of all SM fermions of a single family as

$$S = \left\{ u_L^i, d_L^i, u_R^i, d_R^i, e_L, e_R, \nu_L \right\} \tag{1}$$

where $i = R, G, B$, the color quantum number of the quarks

(i) QCD generators are normalised usually as $\tilde{T}^i = \frac{1}{2} g_s \lambda^i$ where λ^i are the Gell-Mann matrices. Consider the generator

$$\tilde{T}^3 = \frac{1}{2} g_s \lambda^3 \quad , \quad \lambda_3 = diag(+1, -1, 0) \tag{2}$$

Show that the sum over all particles in S of the squares of the eigenvalues of $\tilde{T}^3$ is equal to

$$4 \cdot \left((\frac{g_s}{2})^2 + (-\frac{g_s}{2})^2 \right) = 2 g_s^2 \tag{3}$$

(ii) $SU(2)_L$ generators are also normalised as $T^i = \frac{1}{2} g \sigma^i$, where σ^i are the Pauli matrices. Show that the sum over all particles in S of the squares of the eigenvalues of T^3 is equal to

$$2 \cdot 3 \cdot (\frac{g}{2})^2 + (\frac{g}{2})^2 + (-\frac{g}{2})^2 = 2 g^2 \tag{4}$$

(iii) The U(1) eigenvalue, Y, of each particle are normalised so that $Y = T_3 - Q$, where $T_3 = \pm 1, 0$ depending on whether the particle is in the upper or lower part of an SU(2) doublet, or whether it is an SU(2) singlet, respectively. Show that the sum over all particles in S of the square of the eigenvalue of the U(1) generator, $g'Y$ is equal to

$$\ldots = \frac{10}{3} g^{2\prime} \tag{5}$$

Requiring that the three sums are equal at and above the unification scale provides the condition

$$g_s(M) = g(M) = \sqrt{\frac{5}{3}}g'(M) \tag{6}$$

Exercise 2 The coupling of a gauge theory depends on the typical scale at which the interaction takes place. The dependence is determined by the Renormalization Group Equation (RGE), that, in general, has the form

$$\mu\frac{dg}{d\mu} = -\frac{g^3}{4\pi^2}\beta_1 + \dots \tag{7}$$

where we ignore higher order terms that appear because of renormalization beyond the one-loop level. Integrate this equation from a low scale μ to the GUT scale M, to get

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(M)} - \frac{\beta_1}{2\pi^2}\log(\frac{M}{\mu}) \tag{8}$$

There is one such equation for each SM coupling, g_s, g, g', with different constants $\beta_{1s}, \beta_{1w}, \beta_{1Y}$. We will use

$$\beta_{1s} = \frac{11}{4} - \frac{n_g}{3} \quad \beta_{1w} = \frac{11}{6} - \frac{n_g}{3} \quad \beta_{1Y} = -\frac{5n_g}{9} \tag{9}$$

where n_g is the number of generations, now known to be equal to 3.

Assuming unification at scale M, i.e. that $g_s(M) = g(M) = \sqrt{\frac{5}{3}}g'(M)$, show that, setting $\mu = m_z$ we can get

$$\log\frac{M}{m_z} = \frac{4\pi}{11e^2(m_z)}\left(1 - \frac{8e^2(m_z)}{3g_s^2(m_z)}\right) \tag{10}$$

Does this results depend on the number of generations, n_g? Finally, assume the RG equation is known one order higher, that is

$$\mu\frac{dg}{d\mu} = -\frac{g^3}{4\pi^2}\beta_1 - \frac{g^5}{(4\pi^2)^2}\beta_2 + \dots \tag{11}$$

Integrate it to find $g(\mu)$.

Reminder: $e = g\sin\theta_W = g'\cos\theta_W$