Particle Physics in the Early Universe (FS 17)

Exercise Series 2

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The cosmological principle: Assume that there exists a set of (time-like) geodesics $\{\gamma_o\}_{o\in O}$ which cover spacetime, meaning that the disjoint union of all these curves equals spacetime. One can think of geodesics as the trajectories of inertial observers in spacetime. Fix one of such intertial observers, $\tilde{o}$: at any point of its trajectory, $\gamma_{\tilde{o}}(\tau_0)$, consider two vectors v, w perpendicular to the tangent to the curve γ at the point, $\gamma'_{\tilde{o}}(\tau_0) \perp v, w$. These two directions, v and w, must look the same to the inertial observer: indeed, the request of isotropy is that there is a symmetry of the metric that maps v in w.

Exercise 1 Let u be the vector field of tangent vectors to the geodesics $\{\gamma_o\}_{o\in O}$, and rescale it so that $u^\mu u_\mu = 1$ and write the metric

$$g_{\mu\nu} = u_\mu u_\nu + h_{\mu\nu}, \tag{1}$$

which can be seen as a relation defining $h_{\mu\nu}$ (Have we used the cosmological principle yet?). Show that

$$h_{\mu\nu} u^\nu = 0. \tag{2}$$

Using this, show that there is a reference frame such that

$$g_{\mu\nu} dx^\mu dx^\nu = d\tau^2 + h_{ij} dx^i dx^j. \tag{3}$$

Exercise 2 We will now see how isotropy constrains h_{ij}. In order to do this, consider the Riemann tensor calculated from h_{ij}, and denote it by ${}^{(3)}R^{ab}{}_{cd}$, where the superscript 3 indicates that here the Riemann tensor is calculated from a metric h_{ij} defined on a three dimensional space.

(i) Consider the following map between vectors in three dimensions

$$v'^i = {}^{(3)}R^{ab}{}_{cd}\epsilon_{abj}\epsilon^{cdi}v^j, \quad i,j = 1,2,3, \tag{4}$$

and consider its matrix representation $L_{ij} = {}^{(3)}R^{ab}{}_{cd}\epsilon_{abi}\epsilon^{cd}{}_j$. Show that L is a symmetric linear map. As a consequence, L admits a basis of orthogonal eigenvectors, and can be put in diagonal form. Use the cosmological principle to argue that all the eigenvalues of L must be the same, that is

$$L_{ab} = k\delta_{ab}. \tag{5}$$

Finally, show that (5) implies

$${}^{(3)}R^{ab}{}_{cd} = k(\delta^a{}_c\delta^b{}_d - \delta^a{}_d\delta^b{}_c), \tag{6}$$

and that the curvature scalar is constant for such Riemann tensors. (Remember that $\epsilon^{ijk}\epsilon_{abk} = \delta^i{}_a\delta^j{}_b - \delta^i{}_b\delta^j{}_a$)

(ii) Show that, if

$$h_{ij} = \begin{cases} d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2) \\ d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\phi^2) \\ dx^2 + dy^2 + dz^2 \end{cases}, \tag{7}$$

then (6) is satisfied, and show what conditions are implied on k. One can show, furthermore, that all the metrics satisfying (6) are in this form. This implication was proven by Eisenhart in 1949.

Exercise 3 Finally, change variables to argue that h_{ij} can be sinthetically written as

$$h_{ij} = \frac{1}{1-kr^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{8}$$