Particle Physics in the Early Universe (FS 17)

Exercise Series 2

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September 28, 2017

Exercise 1 For any given metric $g_{\mu\nu}$, the Christoffel symbols, $\Gamma^{\mu}_{\nu\rho}$ satisfy

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\vartheta_\nu g_{\sigma\rho} + \vartheta_\rho g_{\sigma\nu} - \vartheta_\sigma g_{\nu\rho} \right) \tag{1}$$

Show that, for a general Robertson-Walker type metric, that is, any metric of an isotropic homogeneous space, defined by

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a(t)^2 \gamma_{ij} dx^i dx^j \tag{2}$$

where

$$\gamma_{ij} = h(r) dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{3}$$

and

$$h(r) = \frac{1}{1 - \tilde{k} r^2} \quad , \quad \tilde{k} \equiv \frac{k}{L^2} \tag{4}$$

(i) the Christoffel symbols with two or more indices equal to zero vanish:

$$\Gamma^0_{0i} = \Gamma^0_{00} = 0 \tag{5}$$

(ii) the Christoffel symbols with one index equal to zero are given by

$$\Gamma^0_{ij} = a\dot{a}\gamma_{ij} \tag{6}$$

and

$$\Gamma^i_{0j} = \frac{\dot{a}}{a} \delta^i_j \tag{7}$$

Reminder: $\dot{a} \equiv \frac{da(t)}{dt}$

(iii) the Christoffel symbols with all indices different than zero are independent of $a(t)$. They are identical with the Christoffel symbols of the metric γ_{ij}:

$$\Gamma^i_{jk} = \frac{1}{2} \gamma^{im} \left(\vartheta_j \gamma_{mk} + \vartheta_k \gamma_{mj} - \vartheta_m \gamma_{jk} \right) \tag{8}$$

Exercise 2 Let's look at the Christoffel symbols with space indices. Show by direct computation and using step 3 of exercise 1 that the only non-vanishing Christoffel symbols are

$$\Gamma^r_{rr} = \tilde{k}\, r h(r) \tag{9}$$

$$\Gamma^i_{rj} = \frac{1}{r}\delta^i_j \quad , \quad i,j \in \{\theta,\phi\} \tag{10}$$

$$\Gamma^r_{ij} = -\frac{r}{h(r)}\tilde{\gamma}_{ij} \quad , \quad i,j \in \{\theta,\phi\} \quad \tilde{\gamma}_{ij} = diag(1,\sin^2\theta) \tag{11}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \tag{12}$$

$$\Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta} \tag{13}$$

Exercise 3 The Ricci tensor is defined as

$$R_{\nu\rho} = \vartheta_\mu \Gamma^\mu_{\nu\rho} - \vartheta_\nu \Gamma^\mu_{\mu\rho} + \Gamma^\mu_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\mu_{\nu\sigma}\Gamma^\sigma_{\mu\rho} \tag{14}$$

(i) Using the results of the previous exercise, show that

$$R_{00} = -3\frac{\ddot{a}}{a} \tag{15}$$

and

$$R_{ij} = \left(a\ddot{a} + 2\dot{a}^2 + 2\tilde{k}\right)\gamma_{ij} \tag{16}$$

(ii) Compute the curvature scalar $R \equiv g^{\mu\nu}R_{\mu\nu}$ and show that the left-hand side of Einstein's equations for the $0-0$ component is

$$R_{00} - \frac{1}{2}R\, g_{00} = 3\left(\frac{\dot{a}^2}{a^2} + \frac{\tilde{k}}{a^2}\right) \tag{17}$$