Particle Physics in the Early Universe (FS 17)

Exercise Series 5

Dr. Achilleas Lazopoulos

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We will consider a scalar field in curved space-time, described by a Lagrangian density $\mathcal{L}(\phi(x))$. We have seen in exercise sheet 1 that the correct generalization of the action in curved space-time, that is invariant under general coordinate transformations, is

$$I = \int d^4x \sqrt{-g} \mathcal{L}(\phi(x)) \tag{1}$$

where $g \equiv det(g_{\mu\nu})$

The energy-momentum tensor[1] is then

$$T_{\mu\nu} = \frac{\vartheta \mathcal{L}}{\vartheta \vartheta_\mu \phi} \vartheta_\nu \phi - g_{\mu\nu} \mathcal{L} \tag{2}$$

We will consider a spatially homogeneous scalar field: $\phi(x) = \phi(t)$ and therefore all space derivatives vanish, $\vartheta_i \phi = 0$. We will also consider the underlying space-time to be spatially flat, i.e. the metric will take the form

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \tag{3}$$

Exercise 1 Show that the equation of motion for the scalar field, obtained by varying the action, is

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{4}$$

Note that this is the equation of a one-dimensional system[2] in which a particle moves in a potential $V(\phi)$. The expansion of the Universe provides an extra friction term $3H\dot{\phi}$.

[1] The energy-momentum tensor is the conserved Noether current corresponding to the invariance of the Lagrangian $\mathcal{L}$ under translations $x^\mu \to x^\mu - a^\mu$ (there are four conserved four-currents, one for each value of μ).

[2] Not so strange since we got rid of the other three spatial dimensions explicitly...

Exercise 2 The pressure and energy density are related to the energy-momentum tensor by $T_{\mu\nu} = (\rho + P)U_\mu U_\nu - P g_{\mu\nu}$ with $U_\mu = (1, \vec{0})$. Show that

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{5}$$

$$P = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{6}$$

Exercise 3 Show that the Friedmann equation, in such a flat space-time and ignoring any contribution from other fields is

$$H^2 = \frac{8\pi}{3M_{PL}^2}(\frac{1}{2}\dot{\phi}^2 + V(\phi)) \tag{7}$$

Exercise 4 If $\frac{\dot{\phi}^2}{2} \ll V$ then $\rho = -P$ which is the equation of state for a cosmological constant type of matter. This is one of the conditions for the slow-roll inflation scenario. Another condition $\ddot{\phi} \ll 3H\dot{\phi}$, guarantees that the value of the field ϕ changes slowly. Show that in this approximation the dynamics of the Hubble constant are purely determined by the potential and use this and the equation of motion to show that

$$\frac{\dot{H}}{H^2} \approx \frac{3M_{PL}^2}{8\pi}\frac{V'^2}{V^2} \tag{8}$$

Exercise 5 Show that, using the equations of motions and the expression for the Hubble constant in the slow roll approximation it is possible to write

$$\epsilon = \frac{\dot{\phi}^2}{V} \approx \frac{M_{PL}^2}{48\pi}\left(\frac{V'}{V}\right)^2 \ll 1. \tag{9}$$

Furthermore, by differentiating the equations of motions in the slow roll approximation and using the results of the previous exercise, show that

$$\frac{\ddot{\phi}}{3H\dot{\phi}} \approx \frac{M_{PL}^2}{24\pi}\frac{V''}{V} - 2\epsilon \ll 1 \tag{10}$$

Thus the slow roll conditions can be defined in terms of ratios of the derivatives of the potential and the potential himself. What does this imply on the Hubble constant itself and the dynamics of the universe?