Particle Physics in the Early Universe (FS 17)

Exercise Series 10

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December 20, 2019

Exercise 1 For a non-relativistic particle of mass m_X show that the number density in thermal equilibrium at temperature $T \ll m_X$ is

$$n_X = g_X \left(\frac{m_X T}{2\pi}\right)^{3/2} e^{-m_X/T} \tag{1}$$

reminder: the number density is given by

$$n_i = g_i \int \frac{d|p|}{(2\pi)^3} 4\pi |p|^2 \frac{1}{e^{E(p)/T} \pm 1} \tag{2}$$

where $|p|$ is the magnitude of the three-momentum vector that corresponds to the mode with energy $E(p) = \sqrt{|p|^2 + m_X^2}$.

hint: use that

$$\int_m^\infty dE E(E^2 - m^2)^{1/2} e^{-E/T} = m^2 T B_2(\frac{m}{T}) \tag{3}$$

where $B_k(x)$ is the modified Bessel function of the second kind. For $x \gg 1$ we can use the approximation

$$B_2(x) = e^{-x}\sqrt{\frac{\pi}{2x}}\left(1 + \mathcal{O}(1/x)\right) \tag{4}$$

Exercise 2 If particles X and their antiparticles $\bar{X}$ are in thermal equilibrium, the rate for their annihilation is $\Gamma_{XX} = n_x \sigma_0$ where σ_0 is the annihilation cross-section. The annihilation switches off when the temperature drops to the value T_f at which the rate of annihilation is equal to the expansion rate of the Universe (this is the 'freeze-out' temperature). Assuming this happens during the radiation dominated era, when $H(T_f) = \frac{T_f^2}{M_{pl}^*}$, show that

$$\frac{m_X}{T_f} = \log\left(A \cdot (\frac{m_X}{T_f})^{1/2}\right) \tag{5}$$

where

$$A = \frac{\sigma_0 g_X M^*_{pl} m_X}{(2\pi)^{3/2}} \tag{6}$$

If we estimate $\sigma_0 \sim \frac{1}{m_X^2}$ then $A \sim \frac{M^*_{pl}}{m_X}$ is orders of magnitude greater than $\frac{m_X}{T_f}$. Convince yourself that, therefore, we can approximate

$$\frac{m_X}{T_f} = \log(A)\,. \tag{7}$$

Exercise 3 From $\Gamma_{XX} = n_x\sigma_0 = H(T_f)$ we have that the density of the particles X at the moment of freeze-out, t_f is

$$n_X(t_f) = \frac{T_f^2}{M^*_{pl}\sigma_0} \tag{8}$$

Show that their number density today is (hint: their number density in comoving volume does not change after the freeze-out)

$$n_X(t_0) = \frac{a(t_f)^3}{a(t_0)^3} n_X(t_f) \tag{9}$$

Exercise 4 As the Universe expands adiabatically, the entropy is conserved, hence the entropy density satisfies $s(t)a(t)^3 = const.$. The entropy of non-relativistic particles is negligible compared to the entropy of relativistic particles[1]. The entropy contribution of relativistic particles of species i is

$$s_i = g_i \frac{2\pi^2}{45} T_i^3 \quad : (bosons) \quad , \quad s_i = g_i \frac{7}{8}\frac{2\pi^2}{45} T_i^3 \quad : (fermions) \tag{10}$$

where T_i is the temperature at which they are in equilibrium. In the early Universe, at the moment of the decoupling of the heavy particles of the previous exercise, the temperature is common, and all SM particles are relativistic.

At present, only photons and neutrina contribute to the entropy of the Universe, and they are decoupled with temperatures T_γ and T_ν respectively.

[1]This is not so difficult to see, but would derail as here. One can show, from the first law of thermodynamics, that the entropy density is $s = \frac{P+\rho-\mu n}{T}$ and for non-relativistic particles, both ρ and P are proportional to the number density n. The number density of non-relativistic particles is many orders of magnitude smaller than that of relativistic particles, hence the entropy contribution of the former is very small. See Gobrunov and Rubakov sec. 5.1 - 5.2 for (some) further details.

Use the above information to compute the current energy density of Dark Matter particles (including X and $\bar{X}$ and assuming $\rho = M_x \cdot n$) to be

$$\rho_{DM}(t_0) = \rho_{X+\bar{X}}(t_0) = \frac{\left(2T_\gamma^3 + 6\frac{7}{8}T_\nu^3\right)}{g_*(t_f)} \frac{\log(A)}{M_{pl}^* \sigma_0} \tag{11}$$

hint: in order to calculate the entropy you have to take into account the number of flavours and number of polarizations (helicity or spin) when calculating the number of degrees of freedom g_i. Also, observe that the above equation implicitly defined $g_*(t_f)$.