Particle Physics in the Early Universe (FS 17)

Exercise Series 1

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Exercise 1

To Einstein's equations as quickly as possible (following A. Zee: Gravity in a Nutshell)

(i) General Relativity postulates a connection between matter (or energy) that fills space and the geometry of that space. The metric $g_{\mu\nu}$ is seen as a dynamical quantity that changes from point to point in space-time, according to the corresponding equations of motion. We want to start from a field theoretical approach here, according to which, the equations of motion can be derived from minimizing an action, i.e. the integral over space-time of some Lagrangian density, with respect to varying the dynamical field, i.e. the metric. Let's begin from the action of some field as seen in Quantum Field Theory

$$I = \int d^4x \mathcal{L}(\phi(x)) \tag{1}$$

However, in curved spacetime, one can choose any kind of coordinates to describe it. Hence the action should be invariant under a general coordinate transformation

$$x^\mu \to x'^\mu = S^\mu_\nu x^\mu \tag{2}$$

The Lagranigian density $\mathcal{L}(\phi(x))$ should be a scalar under general coordinate transformations. The same, then, should be true for the integral measure. Show that the measure d^4x is not invariant under the transformation in eq.2. Show, however, that the quantity

$$d^4x\sqrt{-g} \quad , \quad g \equiv det(g_{\mu\nu}) \tag{3}$$

is invariant. Hence we should look for an action of the form

$$I = \int d^4x\sqrt{-g}\mathcal{L}(\phi(x)) \tag{4}$$

(ii) Postulate that the Lagrangian is a scalar function of the 'field', i.e. the metric, and it should contain a term with a two derivatives with respect to the metric, in the same way that the Lagrangian of a field has a 'kinetic' term with two derivatives.

(iii) Gravity is related to curvature, and curvature expresses the failure of commutativity in differentiation. So let's start with a covariant vector B_ρ. Show that the derivative of this object, $\vartheta_\mu B_\rho$ does not transform as a second rank tensor, under eq. 2. Instead, show that we have to define a new derivative, the covariant derivative,

$$D_\mu B_\rho \equiv \vartheta_\mu B_\rho - \Gamma^\sigma_{\mu\rho} B_\sigma \tag{5}$$

and fix the transformation properties of the object $\Gamma^\sigma_{\mu\rho}$ such $D_\mu B_\rho$ transforms as a second rank covariant tensor:

$$\Gamma'^\sigma_{\mu\rho} = (S^{-1})^\beta_\mu (S^{-1})^\gamma_\rho S^\sigma_\alpha \Gamma^\alpha_{\beta\gamma} + S^\sigma_\alpha \frac{\vartheta^2 x^\sigma}{\vartheta x'^\mu \vartheta x'^\rho} \tag{6}$$

Note that the covariant derivative defined here is completely analogous to the covariant derivative defined for gauge fields.

(iv) The non-commutativity of derivatives is expressed by defining the Riemann tensor $R^\sigma_{\mu\nu\rho}$ from

$$[D_\mu, D_\nu] B_\rho \equiv -R^\sigma_{\rho\mu\nu} B_\sigma \tag{7}$$

Show that

$$R^\sigma_{\rho\mu\nu} = \left(\vartheta_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\sigma_{\mu\kappa}\Gamma^\kappa_{\nu\rho}\right) - (\mu \leftrightarrow \nu) \tag{8}$$

Taking into account that (we don't show this here)

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\tau} (\vartheta_\mu g_{\nu\tau} + \vartheta_\nu g_{\rho\tau} - \vartheta_\tau g_{\nu\rho}) \tag{9}$$

note that $R^\sigma_{\rho\mu\nu}$ contains terms with two derivatives, so it's the object we are looking for.

(v) Show that $R^\lambda_{\rho\mu\nu} = -R^\lambda_{\rho\nu\mu}$, $R_{\tau\rho\mu\nu} = -R_{\rho\tau\nu\mu}$, $R_{\tau\rho\mu\nu} = -R_{\mu\nu\tau\rho}$. Hence show that out of the Riemann tensor you can only construct one rank-2 tensor, the Ricci tensor

$$R_{\mu\nu} \equiv g^{\tau\rho} R_{\tau\mu\rho\nu} \tag{10}$$

and one scalar

$$R \equiv g^{\mu\nu} R_{\mu\nu} \tag{11}$$

(vi) We are thus led to the action (called the Einstein-Hilbert action) for empty space-time

$$I_{EH} = \int d^4x \sqrt{-g}\, C R(x) \tag{12}$$

Vary the action with respect to the metric, $g_{\mu\nu}$ to get Einstein's equation in empty space:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{13}$$

Note: when you vary the action and you require $\delta I = 0$, there are three terms inside the integral:

$$\delta I_{EH} = \int d^4x \delta(\sqrt{-g}) C g^{\mu\nu} R_{\mu\nu} + \int d^4x \sqrt{-g} C \delta(g^{\mu\nu}) R_{\mu\nu} + \int d^4x \sqrt{-g} C g^{\mu\nu} \delta(R_{\mu\nu}) \tag{14}$$

The last term is in fact a total derivative, so it does not contribute to the action. If you want to derive this, look at section VI.5 of Zee's book.

Note: You will need the identity $\delta\sqrt{-g} = \sqrt{-g}\frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu}$. If you want to prove this, you need to use $\log(\det M) = \mathrm{Tr}(\log M)$, for $g_{\mu\nu}$ in the place of the matrix M.