Particle Physics in the Early Universe (FS 17)

Exercise Series 3

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The goal of this exercise sheet is to construct a phase diagram of the Universe with the relative densities $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$ along the two axes of the plot. In such a diagram every point corresponds to a possible Universe. Ours is represented by a single point. The diagram can be found at the book by A. Zee, Gravity in a Nutshell, section VIII.2.

Exercise 1 Closed vs Open Universe

We have seen in the course that, at any given moment in time,

$$1 = \Omega_R + \Omega_M + \Omega_\Lambda + \Omega_k \tag{1}$$

where

$$\Omega_k = \frac{\rho_k}{\rho_c} = -\frac{\tilde{k}}{a^2}\frac{3}{8\pi G}\frac{1}{\rho_c} \tag{2}$$

and

$$\rho_c = \frac{3H^2}{8\pi G} \tag{3}$$

We have also seen that Ω_R is negligible today ($\Omega_{R,0} \sim 0$) and decreases further as the Universe expands, so for a discussion about the current and future state of the Universe we will ignore it. Hence we have

$$1 = \Omega_{M,0} + \Omega_{\Lambda,0} + \Omega_{k,0} \tag{4}$$

In the diagram of $\Omega_{\Lambda,0}$ and $\Omega_{M,0}$ draw the line $\Omega_{M,0} + \Omega_{\Lambda,0} = 1$ and convince yourself that all points above the line correspond to closed Universes and below the line to open ones.

Exercise 2 Accelerating vs Decelerating Universe

(i) Write the Friedmann equation as

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - \tilde{k} \tag{5}$$

Now differentiate with respect to time and use the continuity equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P) \tag{6}$$

to arrive at

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) = -\frac{4\pi G}{3}\sum_j(\rho_j + 3P_j) \tag{7}$$

This is, in fact, the equation one gets from the ij components of the Einstein equation (we have only used the 00 component till now). Note that the curvature term is absent here.

(ii) We have seen that the Universe is filled by three different types of fluid: matter, radiation and vacuum energy (aka cosmological constant, Λ, or Dark Energy). Each has its own equation of state (EoS). We can summarize what we saw in the course by writing a general EoS:

$$P_j = w_j \rho_j \tag{8}$$

with $w_R = 1/3$, $w_M = 0$ and $w_\Lambda = -1$.

Use this equation to show that

$$Q \equiv \frac{a\ddot{a}}{\dot{a}^2} = -\frac{1}{2}\left(2\Omega_R + \Omega_M - 2\Omega_\Lambda\right) \tag{9}$$

(iii) Use the above equation, for $t = t_{now}$ to find the line that separates an accelerating Universe from a decelerating one in the plot of the previous exercise.

Exercise 3 Flow in the cosmic diagram

(i) Starting from the definition of Ω_j

$$\Omega_j = \frac{\rho_j}{\rho_c} = \frac{8\pi G}{3H^2}\rho_j \tag{10}$$

take derivative with respect to time and use the continuity equation, eq.6 and the EoS, eq.8 to show that

$$\dot{\Omega}_j = \Omega_j(-3H(1 + w_j) - 2\frac{\dot{H}}{H}) \tag{11}$$

(ii) Show that

$$\frac{\dot{H}}{H^2} = \frac{1}{H^2}\frac{d}{dt}\frac{\dot{a}}{a} = Q - 1 \tag{12}$$

where Q is the acceleration parameter defined by eq.9.

(iii) Show that the evolution of Ω_j in time depends on the other Ω_i:

$$\dot{\Omega}_j = \Omega_j H\left(-3w_j - 1 + \sum_i(1 + 3w_i)\Omega_i\right) \tag{13}$$

(iv) Neglecting Ω_R, write down the flow equation for Ω_M and Ω_Λ. Show that there are three fixed points in the diagram of the previous exercise (but now considered for generic time) i.e. three points where the rate of change of $\Omega_{M,\Lambda}$ vanish, of which only one is a stable point. Stable points are points for which nearby points in both directions flow to them.

(v) Show that the flow equation for Ω_k is

$$\dot{\Omega}_k = H\Omega_k(2\Omega_R + \Omega_M - 2\Omega_\Lambda) \tag{14}$$

Is $\Omega_k = 0$ a fixed point? Under which conditions it is stable?