Previously on DIS

We calculated $\hat{W}^{NLO} \equiv -g_{\mu\nu} \hat{W}^{\mu\nu} = e_q^2 \frac{\alpha_s}{2\pi} \left\{ P_{qq}^{(0)} \left(-\frac{1}{\varepsilon}\right) \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} + \text{finite} \right\}$

$$\text{with} \quad P_{qq}^{(0)} = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right]$$

in order to find $\hat{F}_2, \hat{F}_1$

$$\frac{\hat{F}_2}{x} = \hat{W} + \frac{12x^2}{Q^2} \hat{W}P$$

$$\hat{F}_1 - \frac{\hat{F}_2}{2x} = -\frac{4x^2}{Q^2} \hat{W}P$$

The "^" denote partonic quantities, that will in general depend on $z = \frac{x}{\xi}$.

These, in turn, we needed because

$$\frac{d^2\hat{\sigma}}{dx\,dQ^2} = \frac{4\pi}{x} \frac{\alpha^2 e_q^2}{Q^4} \left[x y^2 \hat{F}_1 + \left(1 - y - \frac{(xyM)^2}{Q^2}\right) \hat{F}_2 \right]$$

or

$$\cancel{d\sigma = \frac{1}{\text{flux}} [dk'] [dX] \sum_{\text{spins}} \overline{|M|^2} (2\pi)^4 \delta^4(P+k-k'-X)}$$

with $\cancel{|M|^2 = M_{\mu\nu} M^{*\mu\nu} = L_{\mu\nu} W^{\mu *}} W^{\nu *}$

$$d\hat{\sigma} = \frac{1}{\text{flux}} \frac{(e^2 e_q)^2}{Q^4} L_{\mu\nu} \hat{W}^{\mu\nu} [dk'] \, 4\pi.$$

and $\hat{W}^{\mu\nu}$ was written in terms of $\hat{F}_{1,2}$.

Passing to hadronic quantities means multiplying by $q(\xi)$ and integrating over ξ in $[0,1]$.

$$W \equiv \int \frac{d\xi}{\xi} q(\xi) \hat{W}$$

$$F_{1,2} = \int \frac{d\xi}{\xi} q(\xi) \hat{F}_{1,2}$$

$$d\sigma = \int \frac{d\xi}{\xi} q(\xi) d\hat{\sigma}$$

$$\hat{W} = \hat{W}^{LO} + \hat{W}^{NLO} = x e_q^2 \delta(\xi - x) + e_q^2 \frac{\alpha_s}{2\pi} \left[P_{qq}^0(x) \left(-\frac{1}{\epsilon}\right) \left(\frac{4\pi\mu^2}{q^2}\right)^\epsilon + \text{fin} \right]$$

Setting $\xi = \frac{x}{z} \Rightarrow d\xi = -\frac{x}{z^2} dz = -\frac{\xi}{z} dz$

we have $\frac{d\xi}{\xi} = -\frac{dz}{z}$. $\quad \xi = 0 \rightarrow z = 1$

$\quad \xi = 1 \rightarrow z = x$

$$\hat{W} = e_q^2 q(x) + e_q^2 \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} q\left(\frac{x}{z}\right) \left[P_{qq}^0(z) \left(-\frac{1}{\epsilon}\right) \left(\right)^\epsilon + \text{fin} \right]$$

→ We have not calculated

(a) the longitudinal correction WP → finite

(b) the contribution from the gluon ~~case~~ → exercise

$$\frac{F_2}{x} = W + \text{finite} = \dots$$

→ ~~[illegible]~~ Including the gluon means including the gluon-quark splitting function $P_{qg}^{(0)} = 2 T_F \left(z^2 + (1-z)^2 \right)$

$$\frac{F_2}{x e_q^2} = q(x) + \int_x^1 \frac{dz}{z}\, q\left(\frac{x}{z}\right) \frac{\alpha_s}{2\pi} \left[P_{qq}^0 \left(-\frac{1}{\epsilon}\right) + \text{finite} \right].$$

$$+ \int_x^1 \frac{dz}{z}\, g\left(\frac{x}{z}\right) \frac{\alpha_s}{2\pi} \left[P_{qg}^0 \left(-\frac{1}{\epsilon}\right) + \text{finite} \right].$$

We have a remaining collinear singularity, which is of non-perturbative nature. We would like to factorize all non-perturbative contributions to the PDFs. So we define the RENORMALIZED PDF

$$q^F(x) = q(x) + \int_x^1 \frac{dz}{z}\, q\left(\frac{x}{z}\right) \frac{\alpha_s}{2\pi} \left[P_{qq}^0 \left(-\frac{1}{\epsilon}\right) + R_q(z) \right]$$

$$+ \int_x^1 \frac{dz}{z}\, g\left(\frac{x}{z}\right) \frac{\alpha_s}{2\pi} \left[P_{qg}^0 \left(-\frac{1}{\epsilon}\right) + R_g(z) \right]$$

The finite pieces are arbitrary, and ammount to the definition of the "factorization scheme".

In $\overline{MS}$ $R_q = R_g = 0$ (i.e. we subtract only the pole).

Then $$\frac{F_2}{x e_q^2} = q^F(x) + \int_x^1 \frac{dz}{z}\, q\left(\frac{x}{z}\right) \frac{\alpha_s}{2\pi} \left[P_{qq}^0 \log\left(\frac{Q^2}{\mu_F^2}\right) + \text{finite} \right]$$

$$+ \int_x^1 \frac{dz}{z}\, g\left(\frac{x}{z}\right) \frac{\alpha_s}{2\pi} \left[P_{qg}^0 \log\left(\frac{Q^2}{\mu_F^2}\right) + \text{finite} \right]$$

Note that F_2 ~~and q^F are~~ is supposed to be independent of μ_F : F_2 is physical ~~and q^F is $q(x)$ minus the pole~~.

But when truncating the Pert. Series you actually have a μ_F dependence.

$$\mu^2 \frac{d}{d\mu^2} \frac{F_2}{x e_q^2} = \mu^2 \frac{d q^F}{d\mu^2} + \int_x^1 \frac{dz}{z}\, q\left(\frac{x}{z}\right) \frac{\alpha_s}{2\pi} \left[P_{qq}^{(0)} \frac{\mu^2 d}{d\mu^2} \log\frac{Q^2}{\mu^2} \right]$$

$$+ \dots$$

$$\Rightarrow \quad \mu^2 \frac{d q^F}{d\mu^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} \left[q\left(\frac{x}{z}\right) P_{qq}^{(0)} + g\left(\frac{x}{z}\right) P_{qg}^{(0)} \right]$$

→ Generalization to all orders

$$q(x),\ \bar{q}(x),\ g(x) \equiv f_i(x) \quad i = q, \bar{q}, g$$ for many flavors.

$$\boxed{\mu^2 \frac{\partial f_i(x)}{\partial \mu^2} = \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \sum_j P_{ij}\, f_j\left(\frac{x}{z}\right)}$$ DGLAP equations

$$\underbrace{P_{ij}} = P_{ij}^{(0)} + \frac{\alpha_s}{2\pi} P_{ij}^{(1)} + \dots$$

Altarelli-Parisi splitting function

We don't know $f_i(x, \mu_F)$ but we can compute $f_i(x, \mu')$ from $f_i(x, \mu)$ by evolution!

The DGLAP equations are coupled. It is in general difficult to solve them.

To demonstrate a simple case, consider

$$q^{NS} \equiv q(x) - \bar{q}(x).$$

Then, because $P_{qg} = P_{\bar{q}g}$, the q^{NS} is independent of the gluon PDF. So its DGLAP equation is decoupled

$$\mu^2 \frac{\partial q^{NS}}{\partial \mu^2} = \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} q^{NS}\left(\frac{x}{z}\right) P_{qq}^{(0)}$$

We will now perform a Mellin transformation

$$q(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dn\; x^{-n}\,\tilde{q}(n)$$

One can go back to $q(x)$ via $\tilde{q}(n) = \int_0^1 dx\; x^{n-1} q(x)$.

The DGLAP becomes

$$\mu^2 \frac{\partial}{\partial \mu^2}\; \frac{1}{2\pi i}\int dn\; x^{-n}\,\tilde{q}(n) = \int_x^1 \frac{dz}{z}\,\frac{\alpha_s}{2\pi}\int \frac{dn}{2\pi i}\left(\frac{x}{z}\right)^{-n}\tilde{q}(n)\,P_{qq}^{(0)}(z)$$

$$\frac{1}{2\pi i}\int dn\; x^{-n}\,\mu^2\frac{\partial \tilde{q}(n)}{\partial \mu^2} = \frac{\alpha_s}{2\pi}\int \frac{dn}{2\pi i}\; x^{-n}\tilde{q}(n)\underbrace{\int_x^1 dz\; z^{n-1}\,P_{qq}^{(0)}(z)}_{\gamma(n)}$$

So $\gamma(n) = \int_x^1 dz\; z^{n-1}\,P_{qq}^{(0)}(z)$

and this implies

$$\mu^2\frac{\partial \tilde{q}(n)}{\partial \mu^2} = \frac{\alpha_s}{2\pi}\,\gamma(n)\,\tilde{q}(n).$$

which you can solve to find

$$\tilde{q}(n,\mu) = \tilde{q}(n,\mu_0)\left(\frac{\mu^2}{\mu_0^2}\right)^{\frac{\alpha_s(\mu)\gamma}{2\pi}}$$

note that α_s also depends on μ, and remember $\frac{d\alpha_s}{\alpha_s^2} = -b\,\frac{dQ^2}{Q^2}$

so $$\tilde{q}(n,\mu) = \tilde{q}(n,\mu_0)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\gamma/2\pi b_0}$$

* The anomalous dimension alters the scaling of $\tilde{q}$ with Q^2.

* All information about the evolution is contained in $\gamma(n)$ which is equivalent to the Altarelli-Parisi splitting kernels.

One can evaluate the γ's directly from the splitting functions:

$$\gamma_{qq}(n) = \int_0^1 dz \, P_{qq}(z) = C_F \left[\frac{1}{n(n+1)} + \frac{3}{2} - 2 \sum_{m=1}^{n} \frac{1}{m} \right]$$

$$\gamma_{qg}(n) = \frac{1}{2} \left[\frac{2+n+n^2}{n(n+1)(n+2)} \right]$$

etc.

→ note $\gamma_{qq}(1) = C_F \left[\frac{1}{2} + \frac{3}{2} - 2 \right] = 0$

Generalization / DIS phenomenology.

$$\frac{F_2}{x} = \sum_i e_q^2 \, q(x)$$

Measuring DIS inclusive cross-section → measuring F_2 → measuring $\sum_i e_q^2 q(x)$

For the proton $\sum_i e_q^2 q(x) = \frac{4}{9} u(x) + \frac{1}{9} d(x)$

How to find $u(x), d(x)$ separately?

Assume SU(2) isospin symmetry: proton is neutron with $u \leftrightarrow d$.

Then $F_2^p = \frac{4}{9} u_p(x) + \frac{1}{9} d_p(x)$

$$F_2^N = \frac{4}{9} u_n(x) + \frac{1}{9} d_n(x) = \frac{4}{9} d_p(x) + \frac{1}{9} u_p(x)$$

$\Rightarrow$ F_2^N, F_2^P give info for $u_p(x)$ $d_p(x)$ separately!

→ Definitely more up than down

But $u(x), d(x)$ blows up at $x \to 0$!

So $\int^1 dx\, u(x) = \infty$!

As $x \to 0$, more and more sea quarks appear.

So $F_2^p = \frac{4}{9} u_p(x) + \frac{4}{9} \bar{u}_p(x) + \frac{1}{9} d_p(x) + \frac{1}{9} \bar{d}_p(x)$

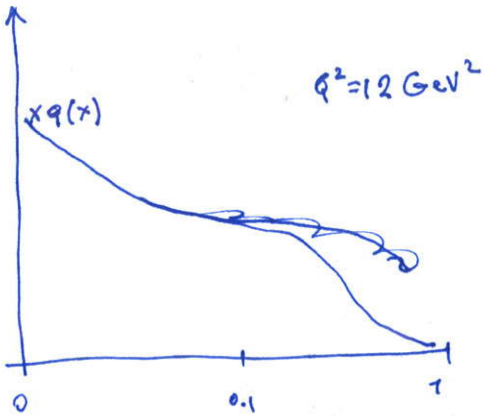
The number of extra pairs can be infinite as long as they carry little momentum!

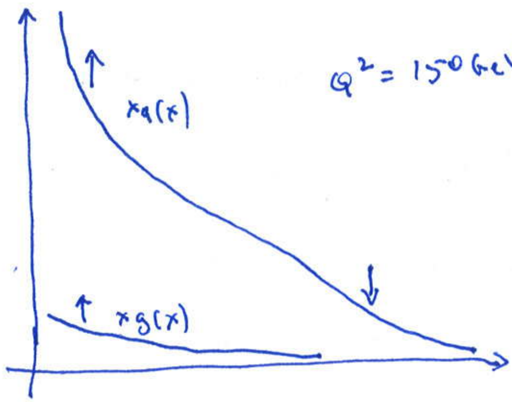
→ When we say the proton has 2 up quarks and one down we really mean that $\int dx \left[u(x) - \bar{u}(x)\right] = 2$

$$\int dx \left[d(x) - \bar{d}(x)\right] = 1.$$

How do we measure the difference between $u(x)$ and $\bar{u}(x)$? The photon interacts identically with $u(x)$ and $\bar{u}(x)$. What interacts differently? W^+/W^-.

Evolution (theory)





Data: if you don't include g(x) distribution in the evolution, the xq(x) doesn't grow fast enough at low x to describe the data!

⇒ The gluon distribution is HUGE. Momentum sums up to 1 now, though!

How do we get the PDFs ?

Various groups publish PDF sets (i.e. $F_i(x, Q^2)$). They do what is called global analysis: they select data from different measurements of various processes, mostly at low Q^2 and for $x \sim 0.1 - 0.01$, they write a parametric expression for F (like $F = x^\alpha (1-x)^\beta$) and they try to fit the parameters to the data (i.e. you guess a values for the parameters, you evaluate the observable, you check with data and you readjust the parameters).

Popular sets come from MSTW, CTEQ, ABKM, NNPDF Delgado-Reyes, etc.

DGLAP evolution very important for LHC.

Points that differ

- data selection
- factorization scheme
- theory input
- value of $\alpha_s(M_Z)$
- error propagation