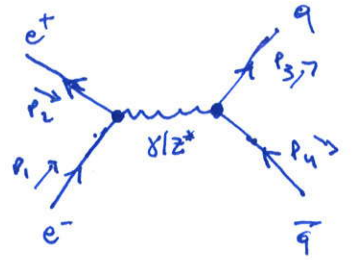
$e^+e^- \to$ hadrons

① Tree-level



$$d\sigma_B = \frac{1}{2E_1 2E_2 |v_1 - v_2|} [dp_3][dp_4] \overline{|M|^2} (2\pi)^4 \delta^4(p_{12} - p_{34})$$

$$[dp_i] = \frac{d^3p_i}{(2\pi)^3 2E_i}$$

Ignoring the Z we have $\quad \sigma_B = \int d\sigma_B = \dots = \frac{4\pi\alpha^2}{3s} N_c \sum_i Q_i^2$

The cross-section for $e^+e^- \to \mu^+\mu^-$ is just $\sigma_B = \frac{4\pi\alpha^2}{3s}$. The ratio, defined as $\quad R = \frac{\sigma_{e^+e^- \to \text{hadrons}}}{\sigma_{e^+e^- \to \mu^+\mu^-}} \quad$ is at LO $\quad R_{LO} = N_c \sum_i Q_i^2$

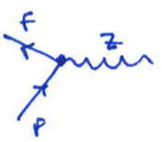
note: The types of quarks one has to take into account are just the ones that have mass (much) lower than $\sqrt{s}$. So if $\sqrt{s} < m_c$ then

$$\sqrt{s} < m_c \simeq 1\,\text{GeV} \qquad \sum_i Q_i^2 = Q_u^2 + Q_d^2 + Q_s^2 = \frac{4}{9} + \frac{1}{9} + \frac{1}{9} = \frac{6}{9} = \frac{2}{3}$$

$$m_c < \sqrt{s} < m_b \sim 4\,\text{GeV} \qquad \sum_i Q_i^2 = Q_u^2 + Q_d^2 + Q_s^2 + Q_c^2 = \frac{10}{9}$$

$$m_b < \sqrt{s} < m_Z \qquad \sum_i Q_i^2 = \dots = \frac{11}{9}$$

note: As one approaches the Z pole $(\sqrt{s} \to m_Z)$ the contribution of Z^* increases in importance. "Remember" that Z couples to fermions via



$= V_f \gamma^\mu + A_f \gamma^\mu \gamma_5$ where V_f, A_f depend on wether the fermion is up or down in its EW doublet, etc.

Close to $\sqrt{s} = m_Z \quad R_Z = 20.09 \quad$ LEP measured $\quad 20.79 \pm 0.04$

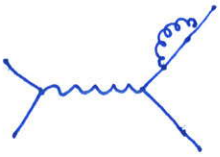
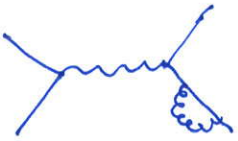
3.5% higher!

2. The renormalization effect.

2. $e^+e^- \to$ hadrons (a) NLO :

$$\sigma^{NLO}_{e^+e^-\to h} = Z_2^2 \sigma_B + \sigma_V + \sigma_R$$

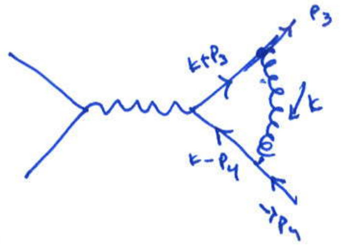
$$\sigma_V = \frac{1}{Flux} \cdot [dP_3][dP_4]\, 2\mathrm{Re}\left\{ M_{Loop} \times M_{tree} \right\} \cdot (2\pi)^4 \delta^4(\quad)$$

$M_{Loop} =$  +  + 

The bubbles are massless with the momentum on-shell, therefore scaleless.

They are actually zero. ~~the com~~ ~~there~~ But there is δZ_2 : $Z_2 = 1 + \delta Z_2$ with δZ_2 cancelling the $\frac{\Sigma(\not p)}{\not p}$ coming from . You will see in the exercises that  $= \frac{g^2}{(4\pi)^2} C_F \not p \left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon'} \right)$. The tricky part is that one of the divergences is UV and the other is of IR nature $(k^2 \to 0)$. So δZ_2 is cancelling the UV's of the bubbles, and the IR is to be added to the amplitude. But so is the finite piece of δZ_2. The net result is of course zero!

3. The virtual triangle



$$= e^2 Q_i\, g_s^2 \; \frac{\bar{u}_2 \gamma^\mu u_1 \; \bar{u}_3 \Lambda_\mu u_4}{s}$$

$$\Lambda^\mu = C_F \int \frac{d^D k}{(2\pi)^D} \; \gamma^\rho \frac{(\not{k}+\not{p}_3)}{(k+p_3)^2} \gamma^\mu \frac{(\not{k}-\not{p}_4)}{(k-p_4)^2} \gamma_\rho \frac{1}{k^2}$$

$$= \dots = C_F \int dx\,dy \int \frac{d^D l}{(2\pi)^D} \frac{A^\mu l^2 + B^\mu}{(l^2-\Delta)^3} \qquad \Delta = -xys$$

$$* \int \frac{d^D l}{(2\pi)^D} \frac{(l^2)^a}{(l^2+\Delta)^b} = \frac{i\,(-1)^{a+b}}{(4\pi)^{D/2}\,\Gamma(D/2)} \frac{\Gamma(D/2+a)\,\Gamma(b-a-D/2)}{\Gamma(b)} \Delta^{D/2+a-b}.$$

the first term is UV divergent: $b=3,\ a=1 \rightarrow \Gamma(b-a-\frac{D}{2}) = \Gamma(2-\frac{D}{2}) = \Gamma(\varepsilon)$

but brings $\Delta^{D/2-2} = \Delta^{-\varepsilon}$

the second term is UV finite: $b=3 \quad a=1 \rightarrow \Gamma(b-a-\frac{D}{2}) = \Gamma(3-\frac{D}{2}) = \Gamma(1-\varepsilon)$

but brings $\Delta^{D/2-3} = \Delta^{-1-\varepsilon}$

Note that $\Delta = -xys$, so the F.P. integral for the second term

will be $\displaystyle\int dx\,dy \frac{B^\mu}{(-s)^{1+\varepsilon}\, x^{1+\varepsilon}\, y^{1+\varepsilon}}$ and B^μ happens to be finite for $x,y \to 0$.

But $\displaystyle\int_0^1 dx \frac{B^\mu}{x^{1+\varepsilon}} \approx \infty$ due to $x \to 0$ blow up.

The divergence is of IR nature and can be traced back at the integral.

It is regularized by ε.

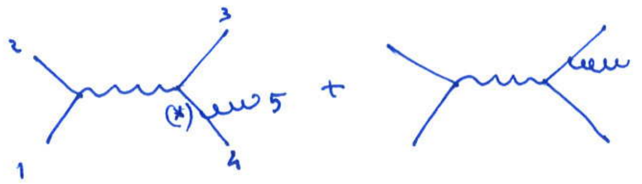
The result for the triangle is

$$\Lambda_\mu = N_C \gamma_\mu \frac{g^2}{8\pi^2} C_F \left(\frac{4\pi\mu^2}{-s}\right)^{\varepsilon} \Gamma(1+\varepsilon)\, B(1-\varepsilon, 2-\varepsilon) \left(\frac{1}{\varepsilon} - \frac{2}{\varepsilon^2} - 2\right)$$

and for the virtual amplitude

$$\sigma_V = A_V \cdot \sigma_B \quad \text{with } A_V = \frac{\alpha_s}{\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\varepsilon} \frac{\cos\pi\varepsilon}{\Gamma(1-\varepsilon)} \left(-\frac{1}{\varepsilon^2} - \frac{3}{2\varepsilon} - 4\right).$$

σ_R :



This amplitude will also have IR divergences! The gluon with momentum p_5 can actually be potentially soft $p_5^\mu \to 0$ or collinear with p_4: $p_5^\mu = \lambda p_4^\mu$ or collinear to p_3^μ: $p_5^\mu = \lambda p_3^\mu$, or both!

For example

$$* : \frac{1}{(p_4+p_5)^2} = \frac{1}{2p_4 \cdot p_5} \simeq \frac{1}{2E_4E_5(1-\cos\theta_{45})}$$

diverges when $p_5 \parallel p_4$ ($\cos\theta_{45} \to 1$) or p_5 is soft ($E_5 \to 0$) or both!

The cross-section is

$$d\sigma_R = \frac{1}{2E_1 2E_2 |v_1 - v_2|} [d\varphi_3][d\varphi_4][d\varphi_5] \; |M_R|^2 (2\pi)^4 \delta^4(P_{12} - P_{345})$$

The phase-space should be worked out in D dimensions also. Then the IR singularities will be regularized as poles in ε! See exercise on 2 and 3 final state particles D-dimensional parametrization.

Note that the matrix element can be parametrized with $x_i = \frac{2p_i \cdot (p_1+p_2)}{s} = \frac{E_i}{\sqrt{s}}$ the energy fraction of the particle "i".

the result for the total real in terms of the σ_B (in D-dimensions) is

$$\sigma_R = A_R \cdot \sigma_B \qquad A_R = \frac{\alpha_s}{\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\varepsilon} \frac{\cos\pi\varepsilon}{\Gamma(1-\varepsilon)} \left(\frac{1}{\varepsilon^2} + \frac{3}{2\varepsilon} + \frac{19}{4}\right)$$

Summing up, one gets for the total cross-section

$$\sigma^{NLO} = Z_2^2\,\sigma_B + \sigma_V + \sigma_R = \sigma_B + \sigma_B A_V + \sigma_B A_R = \sigma_B(1 + A_V + A_R)$$

$$= \left(1 + \frac{3}{4} C_F \frac{\alpha_s}{\pi}\right)\sigma_B$$

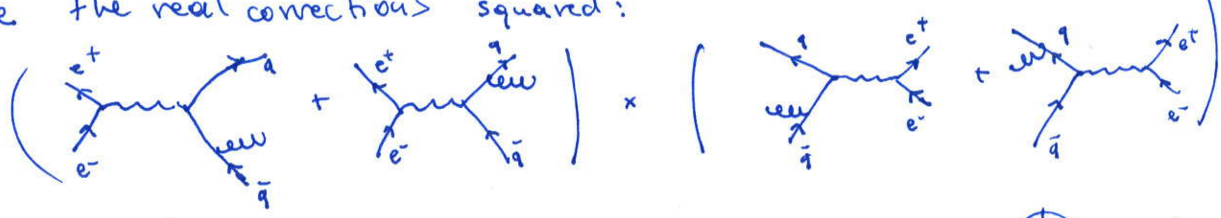
Note: the μ dependence $((\mu^2)^\varepsilon)$ drops out of the result after the poles are cancelled. α_s will of course be μ-dependent.

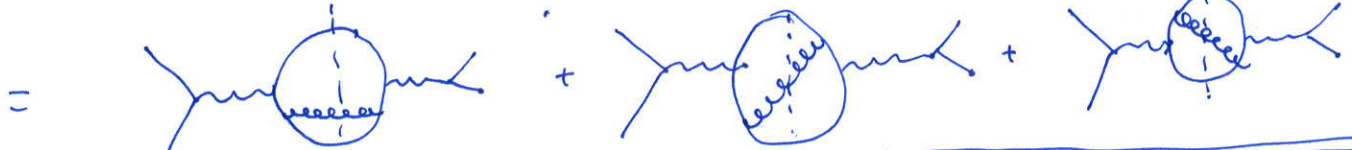
Note: $\frac{\alpha_s}{\pi} \simeq \frac{0.11}{3.14} = 0.035 = 3.5\%$!!!

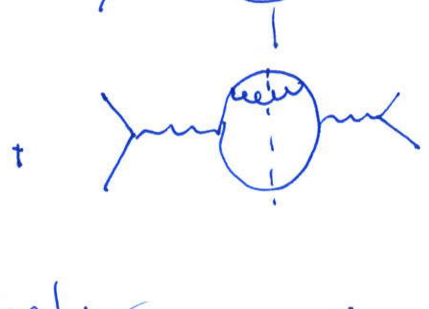
* This is exactly the difference between measured and LO prediction of the theory for R!

This is the pictorial representation of M^*

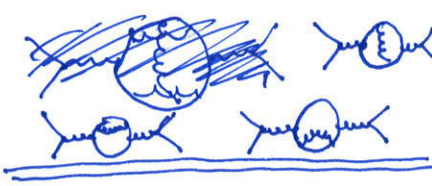
* Let's picture the real corrections squared:



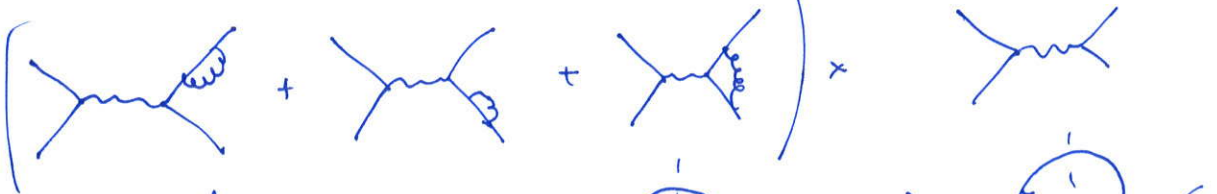


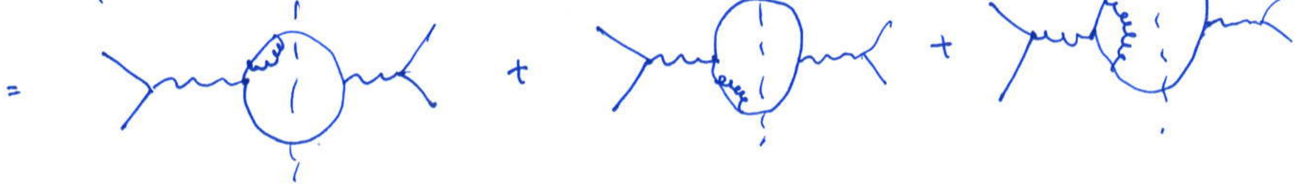


all possible cuts of bubble graphs

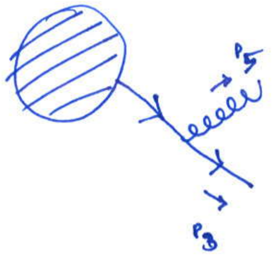


The virtual:





In general, the amplitude for a $n+1$ process at the soft limit simplifies as follows (this is known as the eikonal approximation):

* 

$$\bar{u}(p_3)\,\gamma^\mu \varepsilon_\mu(p_5)\,\frac{\not{p}_5+\not{p}_3}{(p_5+p_3)^2}\,U$$

note: here U stands for everything else in the amplitude (it is a spinor).

$$= \bar{u}(p_3)\,\frac{\not{\varepsilon}_5\,(\not{p}_5+\not{p}_3)}{(p_5+p_3)^2}$$

soft limit $p_5^\mu \to 0$ $\Longrightarrow$ $\bar{u}_3\,\dfrac{\not{\varepsilon}_5\,\not{p}_3}{2p_3\cdot p_5}\,U = \bar{u}_3\,\dfrac{2\varepsilon_5\cdot p_3 - \not{p}_3\not{\varepsilon}_5}{2p_3\cdot p_5}\,U$

$$= \frac{\varepsilon_5\cdot p_3}{p_5\cdot p_3}\,\bar{u}_3\,U = \frac{\varepsilon_5\cdot p_3}{p_5\cdot p_3}\,M_B.$$

note: M_B is the matrix element for the n-particle process

For a given observable O we have at all orders

$$O = \frac{1}{\text{flux}}\sum_n \int [dPS]_n \sum |M_n|^2\, \rho_n(p_i)$$

if $\rho_n = 1$ we get the total cross-section σ

if $\rho_n = \delta(x - x_n(p_i))$ ~~with~~ we get $O = \dfrac{d\sigma}{dx}$

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* In view of the IR cancellations, meaningful physical observables are those that are "IR safe".

e.g. The $e^+e^- \to q\bar{q}$ exclusive cross-section at NLO seems to be infinite.

But the cancelation of IR singularities after you add the $e^+e^- \to q\bar{q}g$ real emission comes from soft-collinear limits.

At those limits, $e^+e^- \to q\bar{q}g$ is actually indistinguishable from $e^+e^- \to q\bar{q}$.

$$\sigma^{NLO}_{\text{Inclusive}, e^+e^- \to \text{hadrons}} = \int [dP_3][dP_4] \left(\sigma_B + \sigma_V\right) + \int [dP_3][dP_4][dP_5] \, \sigma_R$$

exclusive ↓

$$\sigma_{2\text{jets}} = \int [dP_3][dP_4] (\sigma_B + \sigma_V) \, J(P_3, P_4) + \int [dP_3][dP_4][dP_5] \, \sigma_R \, J(P_3, P_4, P_5)$$

with $\lim_{\substack{P_5 \to \text{soft} \\ \text{collinear}}} J(P_3, P_4, P_5) = J(P_3, P_4)$.

$$\sigma_{3\text{jets}} = \int [dP_3][dP_4][dP_5] \, \cancel{J(P_3, P_4, P_5)} \left(1 - J(P_3, P_4, P_5)\right) \sigma_R .$$

$\cancel{J(P_3, P_4, P_5) + \ldots}$

Now note that $\sigma_{2\text{jets}}$ has virtual pieces in it. Virtual can be negative. Real is always positive. By decreasing the parameters in J we artificially decrease $\sigma_{2\text{jets}}$ and increase $\sigma_{3\text{jets}}$. At some point $\sigma_{2\text{jets}}$ will become negative! The dependence is logarithmic on the "cutoff" parameters.