Finite renormalisation

We have seen that starting from a bare Lagrangian, defined in terms of bare parameters m_0, g_0 and α_0, and bare fields A_μ^0, ψ^0, χ^0, one can redefine parameters and fields:

$$m_0 = Z_m \cdot m_R \qquad g_0 = \mu^{\varepsilon} \cdot Z_g \cdot g_R \qquad \alpha_0 = Z_3 \cdot \alpha_R$$

$$A_0^\mu = Z_3^{1/2} A_R^\mu \qquad \psi_0 = Z_2^{1/2} \psi_R \qquad \chi_0 = \tilde{Z}_3^{1/2} \chi_R$$

Such that Green's functions $G_{Rn}(x_1 \dots x_n)$ are finite order by order in perturbation theory.
The procedure of renormalisation leaves a double arbitrariness

1) Subtraction scheme (MS, $\overline{MS}$, on-shell, off-shell ...)

2) Renormalisation scale μ

We now fix a renormalisation procedure, the MS scheme, and investigate what information can be extracted by examining the dependence of renormalised fields and parameters wrt μ.

Simplified case, massless theory, consider g_0 fixed and two different renormalisation scales μ and μ'

$$g_R(\mu) = \mu^{-\varepsilon} Z_g^{-1}(\mu)\, g_0$$

$$g_R(\mu') = (\mu')^{-\varepsilon} Z_g^{-1}(\mu') \cdot g_0$$

which gives $g_R(\mu') = Z_g(\mu';\mu) \cdot g_R(\mu)$ with $Z_g(\mu';\mu) = \dfrac{\mu^{\varepsilon} Z_g(\mu)}{(\mu')^{\varepsilon} Z_g(\mu')}$

We can compose two transformations

$$g_R(\mu'') = z_g(\mu'', \mu')\, z_g(\mu', \mu)\, g_R(\mu)$$

construct the inverse

$$\left[z_g(\mu', \mu)\right]^{-1} = z_g(\mu, \mu')$$

and the identity

$$z_g(\mu, \mu) = 1$$

These operations form an abelian group that is called the renormalisation group (RG). Although we have made use of perturbation theory, these relations can be set up in any renormalisation scheme and beyond perturbation theory.

Analogous relations can be written for all other parameters, in particular for truncated connected Green's functions, defined by

$$(2\pi)^4 \delta^4(p_1 + p_2 + \dots + p_n)\, \tilde{G}_2(p_1)\, \tilde{G}_2(p_2) \dots \tilde{G}_2(p_n)\, \tilde{G}_n^{tc}(p_1 \dots p_n) =$$

$$= \int \prod_{i=1}^{n} d^4 x_i \; e^{-i(p_1 x_1 + p_2 x_2 + \dots + p_n x_n)} \; G_n^t(x_1 \dots x_n)$$

The relation between bare and renormalised Green functions with n_G gluons and n_F quarks is

$$\tilde{G}_{Rn}(x_1 \dots x_n, \mu) = Z_3^{-n_G/2}(\mu)\, Z_2^{-n_F/2}(\mu)\, G_{on}(x_1 \dots x_n)$$

which gives $\tilde{G}_{Rn}^{tc}(p_1 \dots p_n, \mu) = Z_3^{n_G/2}(\mu)\, Z_2^{n_F/2}(\mu)\, \tilde{G}_{on}^{tc}(p_1 \dots p_n)$

or, with finite renormalisations,

$$\tilde{G}_{Rn}^{tc}(p_1 \dots p_n, \mu') = z_n(\mu', \mu)\, \tilde{G}_{Rn}^{tc}(p_1 \dots p_n, \mu)$$

Infinitesimal RG transformation in MS scheme

Suppose we take $\mu' = \mu + \delta\mu$ and we investigate what happens for $\delta\mu \to 0$. Instead of expanding around μ finite RG trasformations, we observe that any bare quantity does not depend on μ. For instance,

$$\tilde{G}^{tc}_{0n}(p_1, \dots p_n, g_0, \alpha_0) = Z_3^{-n_G/2}(\mu)\, Z_2^{-n_F/2}(\mu)\, \tilde{G}^{tc}_{Rn}(p_1 \dots p_n, g_R, \alpha_R, \mu)$$

where we have explicitated the dependence on bare and renormalised parameters for $\tilde{G}^{tc}_{0n}$ and $\tilde{G}^{tc}_{Rn}$ respectively

Using the fact that neither $\tilde{G}^{tc}_{0n}$ nor the bare parameters depend on μ, we have:

$$0 = \mu \frac{d}{d\mu} \tilde{G}^{tc}_{0n}(p_1 \dots p_n, g_0, \alpha_0) =$$

$$= \left[\mu \frac{\partial}{\partial\mu} + \mu \frac{dg_R}{d\mu} \frac{\partial}{\partial g_R} + \mu \frac{d\alpha_R}{d\mu} \frac{\partial}{\partial \alpha_R} - n_G \left(\frac{\mu}{2Z_3} \frac{dZ_3}{d\mu}\right) - n_F \left(\frac{\mu}{2Z_2} \frac{dZ_2}{d\mu}\right)\right]_{g_0, \alpha_0} \cdot$$

$$\cdot \tilde{G}^{tc}_{Rn}(p_1 \dots p_n, g_R, \alpha_R, \mu)$$

We now comment on the various functions

1) Running of the coupling.

From $g_0 = \mu^{\varepsilon} Z_g(\mu)\, g_R(\mu)$ and $\mu \frac{dg_0}{d\mu} = 0$ we find

$$0 = \varepsilon \mu^{\varepsilon} Z_g(\mu)\, g_R(\mu) + \mu^{\varepsilon} \mu \frac{dZ_g(\mu)}{d\mu} g_R(\mu) + \mu^{\varepsilon} Z_g(\mu)\, \mu \frac{dg_R(\mu)}{d\mu}$$

giving

$$\mu \frac{dg_R}{d\mu} = -\varepsilon g_R - g_R \frac{\mu}{Z_g} \frac{dZ_g}{d\mu} = \beta(\varepsilon, g_R)$$

The beta function has the following properties

a) It does not depend on α_R

Proof: g_0 does not depend on α_R, so that $\frac{d}{d\alpha_R}(Z_g \cdot g_R) = 0$ since

$$Z_g = 1 + \frac{Z_g^{(1)}}{\varepsilon} + \frac{Z_g^{(2)}}{\varepsilon^2} + \dots \Rightarrow Z_g \cdot g_R = g_R + \frac{Z_g^{(1)} g_R}{\varepsilon} + \frac{Z_g^{(2)} g_R}{\varepsilon^2} + \dots$$

$$\frac{d\, Z_g \cdot g_R}{d\alpha_R} = \frac{d g_R}{d\alpha_R} + \frac{1}{\varepsilon}\left(\frac{d Z_g^{(1)}}{d\alpha_R} g_R + Z_g^{(1)} \frac{d g_R}{d\alpha_R}\right) + \frac{1}{\varepsilon^2}\left(\frac{d Z_g^{(2)}}{d\alpha_R} g_R + Z_g^{(2)} \frac{d g_R}{d\alpha_R}\right) + \dots$$

This gives $\frac{d g_R}{d\alpha_R} = \frac{d Z_g^{(1)}}{d\alpha_R} = \frac{d Z_g^{(2)}}{d\alpha_R} = \dots = 0$ so that $\frac{d Z_g}{d\alpha_R} = 0$

b) $\beta(\varepsilon, g_R)$ is finite at any order in PT theory for any ε

Proof: consider a RG invariant gauge, like an S-matrix element

$$S_n(p_1 \dots p_n, g_R, \mu) = \left(\sqrt{\frac{Z_3^{LSZ}}{Z_3(\mu^2)}}\right)^n \tilde{G}_{Rn}^{6c}(p_1 \dots p_n, g_R, \mu) =$$

$$= \sqrt{Z_3^{LSZ}} \cdot \tilde{G}_{0n}^{6c}(p_1 \dots p_n)$$

Since S-matrix elements are gauge invariant we have

$$\left[\mu \frac{\partial}{\partial\mu} + \beta(\varepsilon, g_R) \frac{\partial}{\partial g_R}\right) S_n(p_1 \dots p_n, g_R, \mu) = 0$$

$$\beta(\varepsilon, g_R) = - \frac{\mu \frac{\partial}{\partial\mu} S_n(p_1 \dots p_n, g_R, \mu)}{\frac{\partial}{\partial g_R} S_n(p_1 \dots p_n, g_R)} \quad \text{finite}$$

Since $\beta(\varepsilon, g_R)$ is finite we can safely take the limit $\varepsilon \to 0$

This gives us a formula to compute $\beta(\varepsilon, g_R)$ in the MS scheme

$$\beta(\varepsilon, g_R) = -\varepsilon g_R - g_R\, \beta(\varepsilon, g_R) \frac{\partial \ln Z_g}{\partial g_R}$$

$$\beta(\varepsilon, g_R) = \frac{-\varepsilon g_R}{1 + g_R \frac{\partial \ln Z_g}{\partial g_R}}$$

Since the beta function is finite we need to have

$$\left(1 + g_R \frac{\partial \ln Z_g}{\partial g_R}\right)^{-1} = 1 + \frac{Z^{(1)}(g_R)}{\varepsilon}$$

This gives

$$\beta(\varepsilon, g_R) = -\varepsilon g_R + \beta(g_R) \quad \text{where} \quad \beta(g_R) = -Z^{(1)}(g_R)$$

At one loop: $Z_g = 1 + \frac{Z_g^{(1)}}{\varepsilon} g_R^2 + O(g_R^4)$, so that

$$g_R \frac{\partial \ln Z_g}{\partial g_R} = 2 \frac{Z_g^{(1)}}{\varepsilon} g_R^3 \Rightarrow \beta(g_R) = -\beta_0 g_R^3 \qquad \beta_0 = -2 Z_g^{(1)}$$

Using the explicit relation for $Z_g^{(1)}$

$$Z_g = 1 - \frac{g_R^2}{4\pi} \frac{1}{\varepsilon} \cdot \left[\frac{1}{6}(11 C_A - 4 T_R n_f)\right] + O(g_R^4)$$

From which we obtain

$$\beta_0 = \frac{1}{3}(11 C_A - 4 T_R n_f)$$

Exercise: show that, given $Z_g = 1 + \frac{Z^{(1)}(g)}{\varepsilon} + \frac{Z^{(2)}(g)}{\varepsilon^2} + \dots$

$$\beta(g) = g^2 \frac{d Z^{(1)}(g)}{dg}$$

c) The first two coefficient of $\beta(g)$ are <u>scheme independent</u>

$$\beta(g) = -(\beta_0 g^3 + \beta_1 g^5 + \beta_2 g^7 + \dots) \qquad \beta'(g') = -(\beta_0' g'^3 + \beta_1' g'^5 + \beta_2 g'^7 + \dots)$$

g and g' are related by $g' = g + C_1 g^3 + \dots$

Differentiating

$$\beta'(g') = \beta(g) + 3C_1 \cdot g^2 \beta(g)$$

which order by order, becomes

$$\beta_0' g'^3 + \beta_1' g'^5 + \dots = \beta_0 g^3 + \beta_1 g^5 + 3C_1 g^2 (\beta_0 g^3 + \dots) + \dots$$

Up to order g^5 we have, using $g' = g + C_1 g^3$

$$\beta_0' g^3 + (\beta_1' + 3C_1 \beta_0') g^5 = \beta_0 g^3 + (\beta_1 + 3C_1 \beta_0) g^5$$

which gives immediately $\beta_0 = \beta_0'$ and $\beta_1 = \beta_1'$

Solution of 't Hooft-Weinberg equations

Consider the RG equation for $\tilde{G}^{tc}_{Rn}(p, g_R, \alpha_R, \mu)$ with $p = \{p_1 - p_n\}$

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\epsilon, g_R) \frac{\partial}{\partial g_R} + \delta(g_R, \alpha_R) \frac{\partial}{\partial \alpha_R} - n_G \gamma_G(g_R, \alpha_R) - n_F \gamma_F(g_R, \alpha_R)\right] \tilde{G}^{tc}_{Rn}(p, g_R, \alpha_R, \mu)$$

where we have introduced the "anomalous dimensions"

$$\gamma_G(g_R, \alpha_R) = \frac{1}{2} \frac{\mu}{Z_3} \frac{d Z_3}{d\mu}$$

$$\gamma_F(g_R, \alpha_R) = \frac{1}{2} \frac{\mu}{Z_2} \frac{d Z_2}{d\mu}$$

$$\delta(g_R, \alpha_R) = \mu \frac{d\alpha_R}{d\mu} = -2 \alpha_R \gamma_G(g_R, \alpha_R)$$

We now use this RG equation to predict the scaling properties of $\tilde{G}^{tc}_{Rn}$ after a rescaling of momenta $p \to \lambda p$.
By dimensional analysis, if d_G is the mass dimension of $\tilde{G}^{tc}_{Rn}$, we have

$$\tilde{G}^{tc}_{Rn}(\lambda p, g_R, \alpha_R, \mu) = \mu^{d_G} \cdot \phi\left(\frac{\lambda p}{\mu}, g_R, \alpha_R\right)$$

which gives the derivative with respect to μ in terms of $\lambda \frac{\partial}{\partial \lambda}$

$$\mu \frac{\partial}{\partial \mu} \tilde{G}^{tc}_{Rn}(\lambda p, g_R, \alpha_R, \mu) = \left(d_G - \lambda \frac{\partial}{\partial \lambda}\right) \tilde{G}^{tc}_{Rn}(\lambda p, g_R, \alpha_R, \mu)$$

This gives, after defining $t = -\ln \lambda$ $[t \to +\infty$ for $\lambda \to 0]$

$$\left[\partial_t + \beta(g) \frac{\partial}{\partial g} + \delta(g, \alpha) \frac{\partial}{\partial \alpha} + d_G - n_G \gamma_G(g, \alpha) - n_F(g, \alpha)\right] \tilde{G}^{tc}_{Rn}(e^{-t} p, g, \alpha, \mu)$$

where now g and α are just symbols, we thus suppress the index R, and we have set $\epsilon = 0$ since $\beta(\epsilon, g)$ is finite in 4 dimensions

RG equation is solved by constructing the running coupling $\bar{g}(t)$ running gauge parameter $\bar{\alpha}(t)$, defined through the equations

$$\begin{cases} \frac{d}{dt}\bar{g}(t) = \beta(\bar{g}(t)) \\ \bar{g}(0) = g \end{cases} \qquad \begin{cases} \frac{d}{dt}\bar{\alpha}(t) = \delta(\bar{g}(t), \bar{\alpha}(t)) \\ \bar{\alpha}(0) = \alpha \end{cases}$$

An ansatz for the solution is then given by

$$\left[\frac{d}{dt} + \omega_n(\bar{g}, \bar{\alpha})\right] \tilde{G}^{tc}_{Rn}(e^{-t}p, \bar{g}(t), \bar{\alpha}(t), \mu)$$

where $\omega_n(\bar{g}, \bar{\alpha}) = d_G - n_G\gamma_G(\bar{g}, \bar{\alpha}) - n_F\gamma_F(\bar{g}, \bar{\alpha})$

The equation is then easily solved

$$\tilde{G}^{tc}_{Rn}(e^{-t}p, \bar{g}(t), \bar{\alpha}(t), \mu) = e^{-\int_0^t dt'\, \omega_n(\bar{g}(t), \bar{\alpha}(t))} \tilde{G}^{tc}_{Rn}(p, g, \alpha, \mu)$$

We now reverse this relation and get the expression of F_n for large momenta $p \to e^t p$ with $t \to +\infty$

$$\tilde{G}^{tc}_{Rn}(e^t p, g, \alpha, \mu) = e^{\int_0^t dt'\, \omega_n(\bar{g}(t), \bar{\alpha}(t))} F_n(p, \bar{g}(t), \bar{\alpha}(t), \mu) =$$

$$= e^{d_G \cdot t} e^{-n_G\int_0^t dt'\gamma_G(\bar{g}(t), \bar{\alpha}(t))} e^{-n_F\int_0^t dt'\gamma_F(\bar{g}(t), \bar{\alpha}(t))} \tilde{G}^{tc}_{Rn}(p, \bar{g}(t), \bar{\alpha}(t), \mu)$$

In a renormalisable theory $\bar{g}(t) = g$ and $\bar{\alpha}(t) = \alpha$ and $\gamma_G = \gamma_F = 0$

$$\tilde{G}^{tc}_{Rn}(e^t p, g, \alpha, \mu) = e^{d_G t}\tilde{G}^{tc}_{Rn}(p, g, \alpha, \mu)$$

Anomalous dimension correct the naive scaling of function with respect to momentum rescaling

Equation for the running coupling

$$\frac{d\bar{g}(t)}{dt} = \beta(\bar{g}(t)) \Rightarrow t = \int_g^{\bar{g}(t)} \frac{dg'}{\beta(g')}$$

In the PT region the beta function has a zero at $g=0$

$$\beta(g) = -\beta_0 g^3 + \dots \Rightarrow t = -\int_g^{\bar{g}(t)} \frac{dg'}{\beta_0 g'^3} = \frac{1}{2\beta_0}\left(\frac{1}{\bar{g}^2(t)} - \frac{1}{g^2}\right)$$

which gives

$$\bar{g}^2(t) = \frac{g^2}{1+2g^2\beta_0 t} \to 0 \quad \text{for} \begin{cases} t \to +\infty & \text{if } \beta_0 > 0 \\ t \to -\infty & \text{if } \beta_0 < 0 \end{cases}$$

The first case is that of an asymptotically free theory, the coupling constant decreases at large scales (small distances)

Is QCD asymptotically free?

$$\beta_0 = \frac{1}{(4\pi)^2}\,\frac{11 C_A - 4 T_R n_f}{3} > 0 \quad \text{for} \quad n_f < \frac{33}{2}$$

This implies that, for large momenta, we have, owing to the relation

$$\tilde{G}^{tc}_{Ren}(e^t p, g, \alpha, \mu) = e^{d_G t}\, e^{-n_G\int_0^t dt'\,\gamma_G(\bar{g}(t'),\bar{\alpha}(t'))}\, e^{-n_F\int_0^t dt'\,\gamma_F(\bar{g}(t'),\bar{\alpha}(t'))}\, \tilde{G}^{tc}_{Ren}(p, \bar{g}(t), \bar{\alpha}(t), \mu)$$

where on the right-hand side for large t all functions can be expressed as a power series in $\bar{g}(t)$

Note: for $g=0$ $\beta(g)=0$, $g=0$ is a fixed point $\mu\frac{dg}{d\mu}=0$

For $g \neq 0$, the fixed point is approached for $t \to +\infty$, hence $g=0$ is called a UV fixed point

PT theory is

One-loop solution of RG equation for F_n (Exercise)

Consider the explicit expressions for the anomalous dimensions $\gamma_G(g,\alpha)$ and $\gamma_F(g,\alpha)$, as well as $\delta(g,\alpha)$

$$\gamma_G(g,\alpha) = -\frac{g^2}{(4\pi)^2}\left(\frac{C_A}{2}\left(\frac{13}{3}-\alpha\right)-\frac{4}{3}T_R n_f\right)+O(g^4)$$

$$\gamma_F(g,\alpha) = -\frac{g^2}{(4\pi)^2}C_F\cdot\alpha + O(g^4)$$

$$\delta(g,\alpha) = \frac{g^2}{(4\pi)^2}\alpha\left(C_A\left(\frac{13}{3}-\alpha\right)-\frac{8}{3}T_R n_f\right)+O(g^4)$$

First we consider the equation for $\bar\alpha(t)$

$$\frac{d\bar\alpha(t)}{dt} = \delta(g,\alpha) = g^2(t)\alpha(a+\alpha b)$$

Consider the region in which $\bar\alpha(t)$ is close to zero. Then

$$\int_{\alpha'}^{\bar\alpha(t)}\frac{d\alpha'}{a\alpha'} = \int_0^t dt\, g^2(t) = \int_0^t dt\,\frac{g^2}{1+2\beta_0 g^2 t} = \frac{1}{2\beta_0}\ln(1+2\beta_0 g^2 t) \to \infty$$

for $t\to+\infty$

This is consistent with right-hand side only for $a<0$, equivalently

$$13C_A - 8T_R n_f < 0 \Rightarrow 10 \le n_f \le 16 \to \text{asymptotic freedom}$$

In this region we can solve approximately the RG equations by noting that, for $t\to\infty$

$$\gamma_G(g,\alpha) \simeq +\bar\gamma_{G0}\, g^2 + O(g^4) \qquad \gamma_F(g,\alpha) = O(g^4)$$

$$\tilde F^{tc}_{nn}(e^t p, g,\alpha,\mu) = e^{d_G\cdot t}\cdot e^{-n_G\int_0^t dt'\,\bar\gamma_{G0}\,\bar g^2(t)}\,\tilde G^{tc}_{nn}(p,\bar g(t),\bar\alpha(t),\mu) \simeq$$

$$[t\to+\infty] \qquad \simeq e^{d_G\cdot t}(2\beta_0 g^2 t)^{-n_G\cdot\frac{\bar\gamma_{G0}}{2\beta_0}}\,\tilde G^{tc}_{nn}(p,0,0,\mu)$$

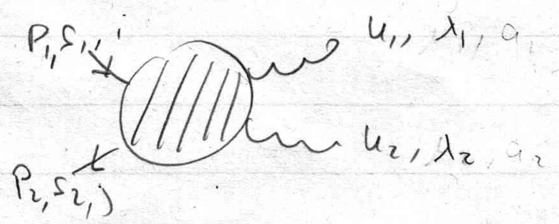
We have then that the naive scaling $e^{d_G t}$ is corrected by logarithmic scaling violations

From Green functions to cross sections

Consider the process $p_1 p_2 \to k_1 \dots k_n$

$$d\sigma = \frac{1}{\text{Flux}} \left| {}_{out}\langle k_1 \dots k_n | p_1, p_2 \rangle_{in} \right|^2 \prod_i \frac{d^3 k_i}{(2\pi)^3 2\omega_i} \qquad \omega_i = k_i^0$$

S matrix elements ${}_{out}\langle k_1 \dots k_m | p_1 \dots p_n \rangle_{in}$ are obtained by using the LSZ formula. Let us specialise to the case of 2→2 scattering, for instance $q\bar{q} \to \gamma\gamma$



$$ {}_{out}\langle k_1, \lambda_1; k_2, \lambda_2 | p_1, s_1; p_2, s_2 \rangle_{in} = \left(\sqrt{Z_3^{LSZ}}\right)^2 \left(\sqrt{Z_2^{LSZ}}\right)^2 \times$$

$$\times\, \varepsilon^{\mu_1}(k_1, \lambda_1)\, \varepsilon^{\mu_2}(k_2, \lambda_2)\, \bar{v}(p_2, s_2) \left[\tilde{G}^{l}_{t,c}(p_1, p_2, k_1, k_2)\right]_{\mu_1, \mu_2} u(p_1, s_1) \Big|_{\substack{p_1^2 = p_2^2 = m^2 \\ k_1^2 = k_2^2 = 0}}$$

$$\times (2\pi)^4 \delta(p_1 + p_2 + k_1 + k_2) \equiv i T(k_1, \lambda_1; k_2, \lambda_2; p_1, s_1; p_2, s_2)$$

where the normalisation constants $\sqrt{Z_3^{LSZ}}$ and $\sqrt{Z_2^{LSZ}}$ are defined in such a way that, on shell

$$\tilde{G}_F(p) \sim \frac{i Z_2^{LSZ}}{\not{p} - m + i\varepsilon} \qquad \tilde{G}_{\mu\nu}(k) \sim \frac{i Z_3^{LSZ}}{k^2 + i\varepsilon} d_{\mu\nu}(k)$$

$(\tilde{G}_{tc}(p_1, p_2, k_1, k_2))_{\mu_1, \mu_2}$ is the truncated connected amplitude obtained from the connected parts of the Green function $\tilde{G}_c(p_1, p_2, k_1, k_2)$ and amputating all external propagators.

Z_3^{LSZ} and Z_2^{LSZ} are renormalisation constants in the <u>on-shell scheme</u>

RG properties of S matrix elements

Neglecting polarisation vectors and constants, the generic form of S matrix elements is

$$S(p_1 - p_n) \sim \left(\sqrt{Z_3^{LSZ}}\right)^{n_G} \left(\sqrt{Z_2^{LSZ}}\right)^{n_F} \tilde{G}^{tc}_{0n}(p_1 - p_n, g_0)$$

1) After renormalisation, S matrix elements are UV finite

$$S(p_1 - p_n) \sim \left(\sqrt{\frac{Z_3^{LSZ}}{Z_3(\mu)}}\right)^{n_G} \left(\sqrt{\frac{Z_2^{LSZ}}{Z_2(\mu)}}\right)^{n_F} \tilde{G}^{tc}_{Rn}(p_1 - p_n, g_R, \mu)$$

Observe that $\tilde{G}^{tc}_{Rn}$ is finite and all UV divergencies cancel in the ratios of renormalisation constants.

2) Being defined in terms of bare quantities and renormalisation constant in the on-shell scheme, S matrix elements are RG invariant. Since they are gauge invariant, they satisfy

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g_R) \frac{\partial}{\partial g_R}\right) S(p_1 - p_n, g_R, \mu) = 0$$

3) The RG invariance $S(p_1 - p_n, g_R, \mu) = S'(p_1 - p_n, g_R', \mu')$ is valid if S is calculated at all orders in PT theory. In general, we can compute S matrix elements up to a given order in g_R, so that

$$S'(p_1 - p_n, g_R', \mu') = S(p_1 - p_n, g_R, \mu) + O(g_R^{n+2})$$

It is then customary to vary renormalisation scale of fixed order predictions around a given value μ, so as to estimate uncertainties due to missing higher orders

4) Choice of the renormalisation scale. RG evolution can help to choose a scale so as to reduce contributions from missing higher orders.

Consider for instance a cross section. Due to Lorentz invariance

$$\sigma(p_1 \ldots p_n, g_R, \mu) = \sigma(\{p_i \cdot p_j\}, g_R, \mu)$$

Let us rescale all momenta by the same quantity $p_i = Q\beta_i$; if d_σ is the mass dimension of σ, we have

$$\sigma(\{p_i \cdot p_j\}, g_R, \mu) = Q^{d_\sigma} \cdot \bar{\sigma}\left(\{\beta_i \cdot \beta_j\}, g_R, \frac{\mu}{Q}\right)$$

Given the RG equation for $\bar{\sigma}$, $\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right)\bar{\sigma}\left(\{\beta_i, \beta_j\}, g, \frac{\mu}{Q}\right) = 0$, defining $t = \ln\frac{Q}{\mu}$, we have

$$\left(\frac{\partial}{\partial t} - \beta(g)\frac{\partial}{\partial g}\right)\bar{\sigma}(\{\beta_i \cdot \beta_j\}, g, \mu e^{-t}) = 0$$

which has the solution $\bar{\sigma}(\{\beta_i \cdot \beta_j\}, g, \mu e^{-t}) = \bar{\sigma}(\{\beta_i \cdot \beta_j\}, \bar{g}(t), 1)$

It is customary to rewrite $\bar{g}^2(t)$ in terms of $\alpha_s(Q^2) = \bar{g}^2(t)/4\pi$

$$\bar{g}^2(t) = \frac{g^2}{1 + 2g^2\beta_0 t} \Rightarrow \alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)\, b_0 \ln\frac{Q^2}{\mu^2}} \qquad b_0 \equiv \frac{\beta_0}{4\pi}$$

Expanding $\alpha_s(Q^2)$ in series of $\alpha_s(\mu^2)$ we get

$$\alpha_s(Q^2) = \alpha_s(\mu^2)\sum_{n=0}^{+\infty}(-1)^n\left(\alpha_s(\mu^2)\, b_0 \ln\frac{Q^2}{\mu^2}\right)^n$$

Consider then the PT expansion of $\bar{\sigma}(\{p_i \cdot p_j\})$

$$\bar{\sigma}(\{p_i\cdot p_j\}) = \underbrace{\bar{\sigma}_0(\{p_i\cdot p_j\})}_{LO} + \alpha_s(Q^2)\underbrace{\bar{\sigma}_1(\{p_i\cdot p_j\})}_{NLO} + \alpha_s(Q^2)\underbrace{\bar{\sigma}_2(\{p_i\cdot p_j\})}_{NNLO} + \dots =$$

$$= \bar{\sigma}_0(\{p_i\cdot p_j\}) + \alpha_s(\mu^2)\bar{\sigma}_1(\{p_i\cdot p_j\}) + \alpha_s^2(\mu^2)\left[\bar{\sigma}_2(\{p_i\cdot p_j\}) - b_0 \ln\frac{Q^2}{\mu^2}\bar{\sigma}_1(\{p_i\cdot p_j\})\right] + \dots$$

with one-loop RG evolution and the knowledge of $\bar{\sigma}_1(\{p_i\cdot p_j\})$, we can predict all terms of the form $\alpha_s^n(\mu^2)\left(\ln\frac{Q^2}{\mu^2}\right)^{n-1}$

a) In order to avoid large logarithms $\ln\frac{Q^2}{\mu^2}$, even if $\alpha_s(\mu^2) \ll 1$, it is useful to fix renormalisation scale Q of the order of momenta p_i.

b) If any $p_i\cdot p_j \ll 1$, RG techniques cannot account for larg $\ln(p_i \cdot p_j)$ appearing in the coefficients σ_i.
One needs to perform an extra resummation of logarithmically enhanced contribution at all orders.

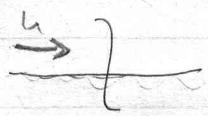
c) Once renormalisation scale Q is fixed, one compares σ to data and extracts $\alpha_s(Q^2)$.
If $\alpha_s(Q^2) \ll 1$, we are in a PT regime, so the QCD calculation is consistent.

Unitarity of S matrix

Slavnov-Taylor identities are fundamental to prove the unitarity of QCD S-matrix.
Perturbative unitarity is the statement that, for a process $i \to f$ with a transition matrix T_{fi}, we have

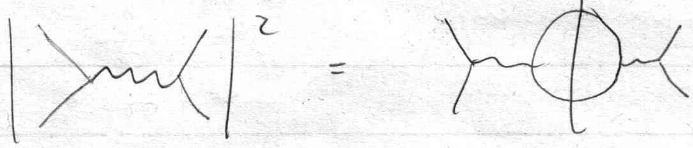
$$2\,\mathrm{Im}\,T_{ii} = \sum_f T^*_{if} T_{fi} = \sum_{\text{cuts}} (T_{ii})_{\text{cut}}$$

where for $(T_{ii})_{\text{cut}}$ we mean that we cut a propagator in such a way that

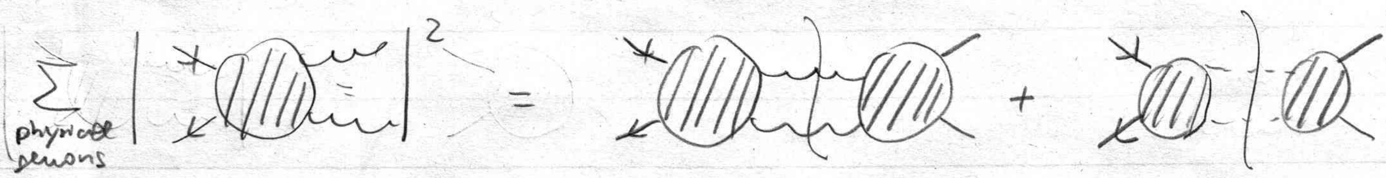


$$\to (2\pi)\,\delta(k^2 - m^2)\,\Theta(k_0)$$

This is equivalent to the statement that we put all particles across the cut on their mass shell. For instance, for $e^+e^- \to \mu^+\mu^-$ we have



In QCD, unitarity is equivalent to saying that, for instance



where in the cut on the RHS non physical polarisation can propagate according to the gauge fixing procedure, but their contribution is cancelled by the ghost contribution. This gives two alternative procedures to compute gluon cross sections

1) Use physical polarisations $\varepsilon \cdot k = \varepsilon \cdot n = n^2 = 0$

$$\sum_{\lambda=1,2} \varepsilon_\mu(k,\lambda)\,\varepsilon^*_\nu(k,\lambda) = -g_{\mu\nu} + \frac{k_\mu n_\nu + k_\nu n_\mu}{n \cdot k}$$

2) Use an arbitrary gauge with all polarisations for external gluon and subtract proper ghost contribution