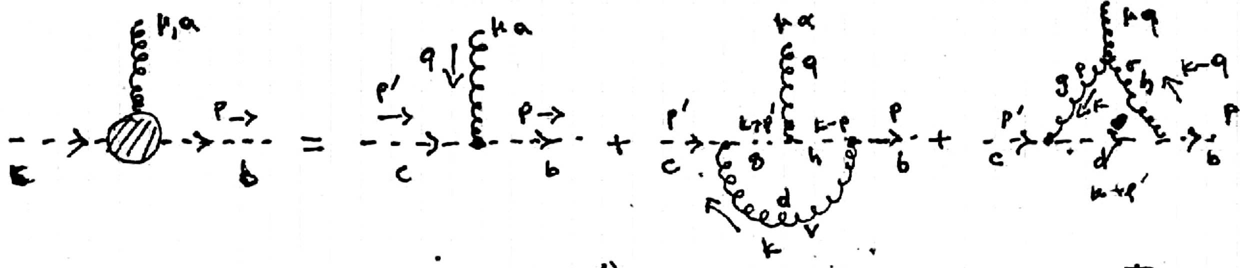


$$-ig\, p_\mu f_{abc}(-i) \quad + \quad I_1 \quad + \quad I_2$$

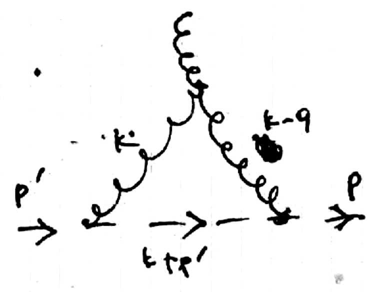
$$I_1 = (+ig)^3 \underbrace{f_{cgd}\, f_{gah}\, f_{hbd}}_{\text{Color factor}} \underbrace{\int \frac{d^Dk}{(2\pi)^D} \frac{(k+p')_\nu (k-p)_\mu\, p_\rho\, (-g^{\nu\rho})}{k^2\,(k-p)^2\,(k+p')^2}}_{\text{integral.}}$$

integral: $$\int \frac{d^Dk}{(2\pi)^D} \frac{(k+p')\cdot p\ (k-p)_\mu}{k^2\,(k-p)^2\,(k+p')^2} = \int \frac{d^Dk}{(2\pi)^D}\, p_\rho \frac{\left(k^\mu k^\rho + k^\mu p'^\rho - p^\mu k^\rho - p^\mu p'^\rho\right)}{k^2\,(k-p)^2\,(k+p')^2}$$

$$= p_\rho \left(-\frac{g_{\mu\rho}}{D}\right) \int \frac{d^Dk}{(2\pi)^D} \frac{k^2}{k^2\,(k-p)^2\,(k+p')^2} \quad + \text{ finite.}$$

$$= -p_\mu \cdot \frac{1}{D} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2\,(k+p'-p)^2} \quad + \text{ finite.}$$

$$= -p_\mu \frac{1}{D} \frac{(4\pi)^\varepsilon\, \Gamma(\varepsilon)}{16\pi^2} \quad + \text{ finite.}$$



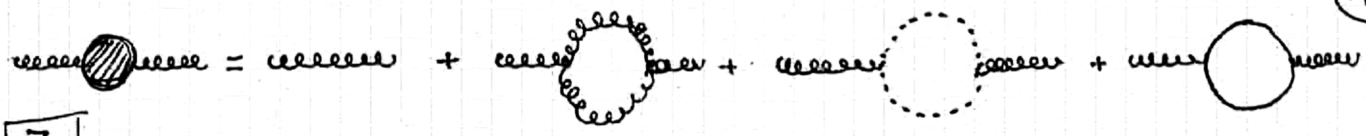
$$I_1 = +g^3 \frac{\text{ColorFactor}}{D} \cdot p_\mu \frac{g^2}{16\pi^2} (4\pi)^\varepsilon\, \Gamma(\varepsilon)$$

$$I_2 = g\, g^2\ \text{ColorFactor} \int \frac{d^Dk}{(2\pi)^D} \frac{(k+p')_\rho\, p_\sigma\, V_{\rho\mu\sigma}(-k, q, k-q)}{k^2\,(k+p')^2\,(k-q)^2}$$

Numerator $$(k+p')_\rho\, p_\sigma \left[ g_{\rho\mu}(-k-q)_\sigma + g_{\mu\sigma}(q-k+q)_\rho + g_{\sigma\rho}(2k-q)_\mu \right]$$

$$= (k+p')_\mu (-k-q)\cdot p + p_\mu (2q-k)\cdot(k+p') + (k+p')\cdot p\,(2k-q)_\mu$$

$$= -k_\mu k_\sigma p^\sigma - k^2 p_\mu + 2k_\sigma k_\mu p^\sigma = \left(+\frac{1}{D} - 1\right) p_\mu k^2$$



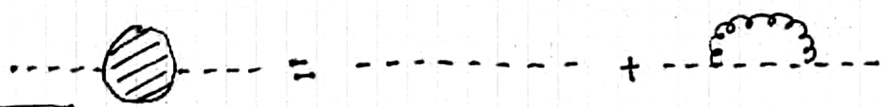
$\boxed{Z_3}$

$$= \frac{-i g_{\mu\nu}\,\delta_{ab}}{Z_3 p^2 + \Pi(p^2)}$$

with $$Z_3 p^2 + \Pi(p^2) = Z_3 p^2 + p^2 \frac{a_s}{4\pi}\left[\frac{4}{3}\cdot\frac{1}{2} n_f - \frac{1}{2} N_c \cdot \frac{10}{3}\right]\left[(4\pi)^\varepsilon \Gamma(\varepsilon)\left(\frac{\mu^2}{-p^2}\right)^\varepsilon + \text{finite}\right]$$

note that $\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + O(\varepsilon)$

In $\overline{MS}$ scheme $$Z_3 = 1 - \frac{a_s}{4\pi}\left[\frac{4}{6} n_f - \frac{10}{6} N_c\right](4\pi)^\varepsilon \Gamma(\varepsilon)$$

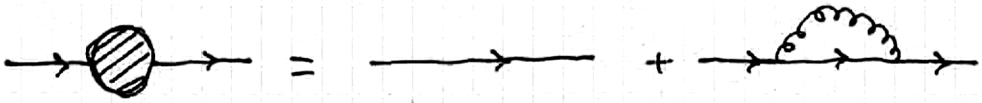


$\boxed{\tilde{Z}_3}$

$$= \frac{i\,\delta_{ab}}{\tilde{Z}_3 p^2 + \tilde{\Pi}(p^2)}$$

with $$\tilde{Z}_3 p^2 + \tilde{\Pi}(p^2) = \tilde{Z}_3 p^2 - p^2 \frac{a_s}{4\pi}\frac{N_c}{2}(4\pi)^\varepsilon \Gamma(\varepsilon)\left(\frac{\mu^2}{-p^2}\right)^\varepsilon + \text{finite}$$

so in $\overline{MS}$ $$\tilde{Z}_3 = 1 + \frac{a_s}{4\pi}\frac{N_c}{2}(4\pi)^\varepsilon \Gamma(\varepsilon)$$



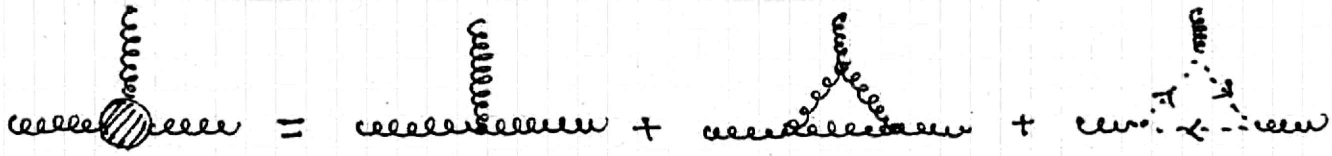
$\boxed{Z_2, Z_m}$

$$= \frac{i\,\delta_{ij}}{Z_2 \not{p} - Z_2 Z_m m + \Sigma(\not{p})}$$

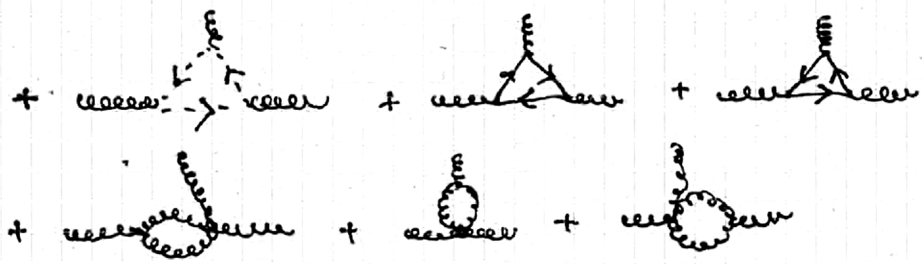
$$Z_2 \not{p} - Z_2 Z_m m + \Sigma(\not{p}) = Z_2 \not{p} - Z_2 Z_m m + \frac{a_s}{4\pi} C_F (4\pi)^\varepsilon \Gamma(\varepsilon)\not{p}$$
$$- \frac{a_s}{4\pi} 4 C_F (4\pi)^\varepsilon \Gamma(\varepsilon)\, m + \text{finite}$$

so in $\overline{MS}$ $$Z_2 = 1 - \frac{a_s}{4\pi} C_F (4\pi)^\varepsilon \Gamma(\varepsilon)$$

$$Z_m = 1 - \frac{a_s}{4\pi} 3 C_F (4\pi)^\varepsilon \Gamma(\varepsilon)$$



$$\boxed{Z_g Z_3^{3/2} \equiv Z_1}$$

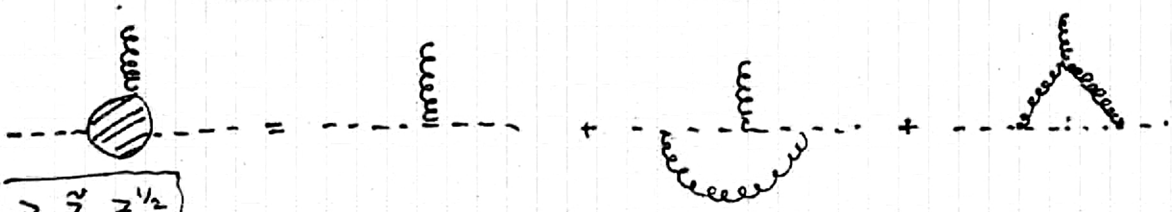


$$= -i\, g\, f^{abc} V_{\mu\nu\sigma}(p_1,p_2,p_3) \left( Z_1 + \Gamma(p_1,p_2,p_3) \right)$$

$$Z_1 + \Gamma(p_1,p_2,p_3) = Z_1 + \frac{a_s}{4\pi} \left( N_c\left(-\frac{2}{3}\right) + \frac{4}{3}\cdot\frac{1}{2} n_f \right)(4\pi)^\varepsilon \Gamma(\varepsilon) + \text{finit}$$

so $$Z_1 = 1 - \frac{a_s}{4\pi} \left[ N_c\left(-\frac{2}{3}\right) + \frac{4}{3}\,\frac{n_f}{2} \right](4\pi)^\varepsilon \Gamma(\varepsilon)$$

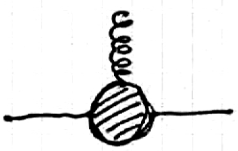
③



$\boxed{\begin{array}{l} Z_g \tilde{Z}_3 Z_3^{1/2} \\ \equiv \tilde{Z}_1 \end{array}}$

$$= -ig\, p_\mu (-if_{abc}) \left( \tilde{Z}_1 + \frac{a_s}{4\pi} N_c \cdot \frac{1}{2} (4\pi)^\varepsilon \Gamma(\varepsilon) + \text{finite} \right)$$

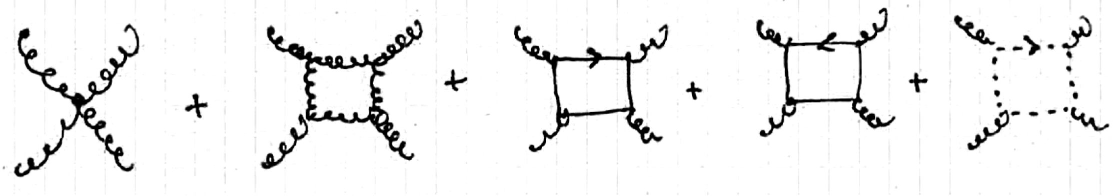
$$\tilde{Z}_1 = 1 - \frac{a_s}{4\pi} N_c \cdot \frac{1}{2} (4\pi)^\varepsilon \Gamma(\varepsilon)$$



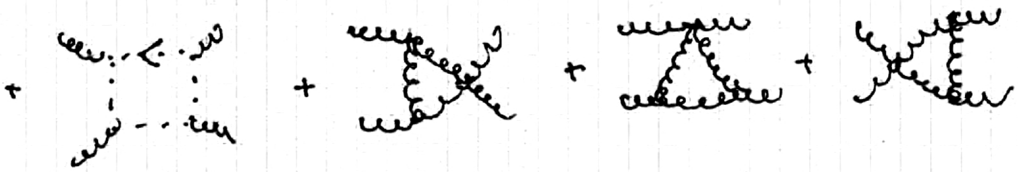
$$= -ig\gamma^\mu t^a_{ij} \left( Z_{1F} + \frac{a_s}{4\pi} (N_c + C_F) (4\pi)^\varepsilon \Gamma(\varepsilon) + \text{finite} \right)$$

$\boxed{\begin{array}{l} Z_g Z_2 Z_3^{1/2} \\ \equiv Z_{1F} \end{array}}$

$$Z_{1F} = 1 - \frac{a_s}{4\pi} (N_c + C_F) (4\pi)^\varepsilon \Gamma(\varepsilon)$$



$\boxed{\begin{array}{l} Z_g^2 Z_3^2 \\ \equiv Z_4 \end{array}}$



$$= -ig^2 V_{4\,\mu\nu\rho\sigma}^{abcd} \left[ Z_4 + \frac{a_s}{4\pi} \left( \frac{N_c}{3} + \frac{4}{3} \frac{n_f}{2} \right) (4\pi)^\varepsilon \Gamma(\varepsilon) + \text{finite} \right]$$

$$Z_4 = 1 - \frac{a_s}{4\pi} \left( \frac{N_c}{3} + \frac{4}{6} n_f \right) (4\pi)^\varepsilon \Gamma(\varepsilon)$$

Defining $\Delta_\varepsilon \equiv \frac{\alpha_s}{4\pi}(4\pi)^\varepsilon \, \Gamma(\varepsilon)$ we have

$$Z_3 = 1 - \left[\frac{2}{3} n_f - \frac{5}{3} N_c\right] \Delta_\varepsilon$$

$$\tilde{Z}_3 = 1 + \frac{N_c}{2} \Delta_\varepsilon$$

$$Z_2 = 1 - C_F \Delta_\varepsilon$$

$$Z_m = 1 - 3 C_F \Delta_\varepsilon$$

$$Z_1 = 1 - \left(N_c\left(-\frac{2}{3}\right) + \frac{2}{3} n_f\right) \Delta_\varepsilon = Z_g Z_3^{3/2}$$

$$\tilde{Z}_1 = 1 - \frac{N_c}{2} \Delta_\varepsilon = Z_g \tilde{Z}_3 Z_3^{1/2}$$

$$Z_{1F} = 1 - (N_c + C_F) \Delta_\varepsilon = Z_g Z_2 Z_3^{1/2}$$

$$Z_4 = 1 - \left(\frac{N_c}{3} + \frac{2}{3} n_f\right) \Delta_\varepsilon = Z_g^2 Z_3^2$$

Note:

$$\frac{Z_1}{Z_3} = Z_g Z_3^{1/2} = 1 - \frac{2}{3} n_f \Delta_\varepsilon + \frac{2}{3} N_c \Delta_\varepsilon + \frac{2}{3} n_f \Delta_\varepsilon - \frac{5}{3} N_c \Delta_\varepsilon$$

$$= 1 - N_c \Delta_\varepsilon$$

$$\frac{\tilde{Z}_1}{\tilde{Z}_3} \overset{Z_g Z_3^{1/2}}{=} 1 - \frac{N_c}{2} \Delta_\varepsilon - \frac{N_c}{2} \Delta_\varepsilon = 1 - N_c \Delta_\varepsilon$$

$$\frac{Z_{1F}}{Z_2} = Z_g Z_3^{1/2} = 1 - N_c \Delta_\varepsilon - C_F \Delta_\varepsilon + C_F \Delta_\varepsilon = 1 - N_c \Delta_\varepsilon$$

$$\frac{Z_4}{Z_1} = 1 - \frac{N_c}{3} \Delta_\varepsilon - \frac{2}{3} n_f \Delta_\varepsilon + \frac{2}{3} n_f \Delta_\varepsilon - \frac{2}{3} N_c \Delta_\varepsilon = 1 - N_c \Delta_\varepsilon$$

$$\Rightarrow \boxed{\frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3} = \frac{Z_{1F}}{Z_2} = \frac{Z_4}{Z_1} = Z_g Z_3^{1/2} = 1 - N_c \Delta_\varepsilon}$$

$$\mathcal{L} = \underbrace{\mathcal{L}_{gluon} + \mathcal{L}_{fermion}} + \mathcal{L}_{\substack{gauge\\fixing}} + \mathcal{L}_{ghost}$$

invariant under $\delta A_\mu^a = -\frac{1}{g} D_\mu^{ab} \theta^b$

$$\delta \psi = -i\, t^a \theta^a \psi$$

$$\text{set } \theta^a = -g\, \delta\lambda\, \chi_2^a \qquad \chi^a = \frac{\chi_1^a + i\chi_2^a}{\sqrt{2}}.$$

~~imaginary part~~

grassman $\{\delta\lambda, \chi_2^a\} = 0$.

$$\left.\begin{aligned} \Rightarrow \quad \delta A_\mu^a &= \delta\lambda\, D_\mu^{ab} \chi_2^b \\ \delta\psi &= \delta\lambda\, ig\, t^a \chi_2^a \psi \\ \delta\chi_1^a &= i\,\delta\lambda\, \frac{1}{\alpha}\, \partial\cdot A^a \\ \delta\chi_2^a &= -\frac{1}{2}\delta\lambda\, g\, f^{abc}\chi_2^b \chi_2^c . \end{aligned}\right\} \text{BRS symmetry.}$$

- Quantized version of QCD Lagrangian includes a gauge-fixing term and a ghost term
- The local gauge invariance transformation is broken by those terms
- Extending the trs. to include ghosts (in a non-trivial way) leads to the BRS transformations that <u>are</u> a symmetry of the Lagrangian.

- In the path-integral formalism, all order expressions for Green's functions are obtained by the generating functional

$$Z[J, \xi_1, \xi_2, \eta, \bar{\eta}] = \int [dA][d\chi_1][d\chi_2][d\psi][d\bar{\psi}]\, e^{i\int d^4x \left(\mathcal{L} + AJ + \chi_1\xi_1 + \chi_2\xi_2 + \bar{\psi}\eta + \bar{\eta}\psi\right)}$$

- Invariance of the Lagrangian and the measure leads to relations between Green functions: The Ward-Takahashi's

- These identities, since they involve Green functions (i.e. objects that include all order corrections), are valid to all orders in perturbation theory and they are used to prove renormalizability of QCD to all orders inductively.
- To one-loop, they lead to the $Z$ relationship that we've shown.
- All orders renormalization means that without adding new terms in the Lagrangian or new renormalization constants, all amplitudes at any given order in perturbation theory can be made finite by adjusting for that order the values of, e.g. $Z_2, Z_3, \tilde{Z}_3, Z_g, Z_m$