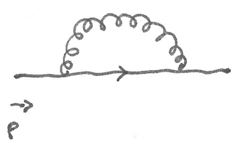
Introduction to QCD — lecture 4: Renormalization of QCD

- Perturbative calculations of physical cross-sections for processes involving quarks and gluons are performed using the usual Feynman diagram formalism. One, here, defers the problems of "factorization" and "hadronization" to some further discussion and assumes the existence of free quarks and gluons (and hence the corresponding assymptotic states.

- Tree-level calculations reproduce some features of the parton model, but all interesting QCD effects appear beyond the leading order (LO).

- The corrections due to higher order diagrams can be quite large, due to the not-so-small value of α_s (~ 0.1).

- Diagrams beyond the LO are divergent!

Example:  $\rightarrow$ $g^2 C_F \int \frac{d^4k}{(2\pi)^4} \frac{(-2)(\not{p}-\not{k})+4m}{k^2\left[(p-k)^2-m^2\right]}$

when $k^\mu \to \infty$ $\qquad g^2 C_F \int \frac{dk}{(2\pi)^4} \frac{k^3\, d\Omega_3 \cdot \not{k}\,(-2)}{k^4}$

UV divergence $\qquad \sim g^2 C_F \int_0^\infty dk \sim \lim_{k\to\infty} k$

note: actually, due to symmetry, the integral is $\sim \int_0^\infty \frac{dk}{k} = \lim_{k\to\infty} \log k$

→ See exercise for details on the calculation.

$\boxed{\text{note}}$: it is the same loop diagram as the one you've seen in QED, multiplied by the color factor $C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3}$

- To handle the divergences, integrals have to be REGULARIZED. We will use $\boxed{\text{DIMENSIONAL REGULARIZATION}}$ in this course.

- "Renormalization" is the procedure through which all divergences arising from Feynman diagrams at all orders are absorbed into a redefinition of fields, masses and coupling constants.

Which diagrams have UV divergences?

POWER COUNTING

- In any given diagram the UV behaviour is found by the limit of loop momenta to infinity.
- Every loop integration $\int \frac{d^D k}{(2\pi)^D} = \int \frac{k^{D-1}\, dk\, d\Omega_D}{(2\pi)^D}$ contributes a factor of D to the degree of divergence of the integral.
- Every three-gluon vertex brings a factor of k^μ in the numerator
- Every quark propagator brings a factor of $\frac{\not{k}}{k^2} \sim \frac{1}{k}$
- Every gluon propagator brings a factor of $\frac{1}{k^2}$
- Every ghost propagator brings a $\frac{1}{k^2}$ ~~in the numerator~~
- Every gluon-ghost-ghost vertex brings $\propto k^\mu$ in the numerator

Defining the "superficial degree of divergence" we have

$$d = D\, n_L + V_3 - P_q - 2P_g - 2P_{gh} + V_{gg\,gh.}$$

D: # of dimensions; n_L: # of loops; V_3: # of ggg vertices; P_q: # of quark propagators; P_g: # of gluon propagators; P_{gh}: # of ghost propagators; $V_{gg\,gh}$: # of gluon ghost ghost vertices.

When $d \geq 0$ the diagram is UV divergent

$d < 0$ the diagram is finite

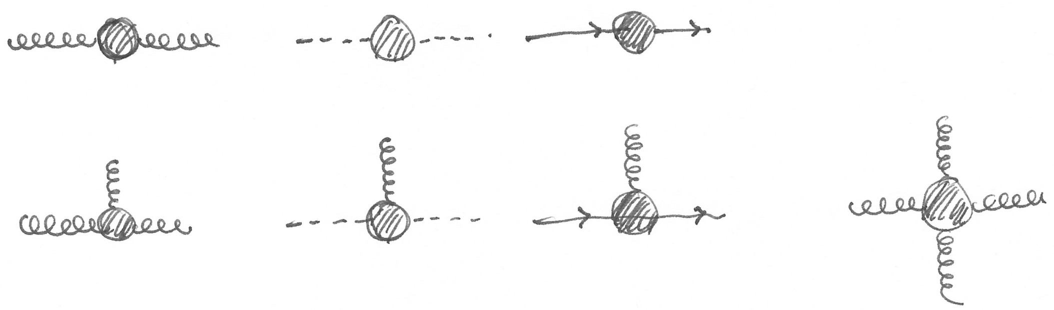
- The counting must be applied <u>on every subgraph</u>.

- The actual behaviour of the corresponding integral might be better due to cancellations coming from symmetries.

- One can apply topological relations (like $2\underbrace{p_q}_{\#\text{ of quark propagators}} + \underbrace{e_q}_{\#\text{ of external quarks}} = 2\overbrace{v_q}^{\#\text{ of } qqg \text{ vertices}}$)

to express d as a function of the external legs of 1PI diagrams.

One finds then that only the following 1PI graphs can be divergent:



→ In other words all divergences of UV nature in QCD appear in one of the above 7 building blocks. If one absorbs all divergences of those building blocks by renormalizing the Lagrangian, the renormalization program is complete and all amplitudes are finite (in the UV).

RENORMALIZED LAGRANGIAN

The QCD Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^2 - \frac{1}{2a} (\partial \cdot A)^2 + \bar{\psi} (i\not{D} - m)\psi + \partial_\mu \chi_a^* \partial^\mu \chi_a - g \partial^\mu \chi_a^* f_{abc} A_b^\mu \chi_c$$

$$F^2 \equiv F^a_{\mu\nu} F^{a\,\mu\nu} \quad \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + ig f^{abc} A_\mu^b A_\nu^c = F^a_{\mu\nu}$$

$$\text{or } F^a_{\mu\nu} = \partial_{[\mu} A^a_{\nu]} + ig f^{abc} A_\mu^b A_\nu^c$$

$$D_\mu = \partial_\mu + ig A_\mu^a t^a$$

We renormalize:

$$A_\mu^a = \sqrt{Z_3}\, A^a_{\mu R} \qquad g = Z_g g_R$$

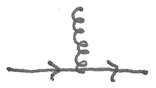
$$\chi_a = \sqrt{\tilde{Z}_3}\, \chi_{aR} \qquad m = Z_m m_R$$

$$\psi = \sqrt{Z_2}\, \psi_R \qquad a = Z_3 a_R$$

$\mathcal{L}^{BARE} = -\frac{1}{4} \partial_{[\mu} A^a_{\nu]} \partial^{[\mu} A^{\nu]a}$	Z_3	
$-\frac{1}{2} ig f^{abc} \partial_{[\mu} A^a_{\nu]} A^{\mu b} A^{\nu c}$	$Z_3^{3/2} Z_g$	$\equiv Z_1$
$-\frac{1}{4} i^2 g^2 f^{abc} f^{adf} A_\mu^b A_\nu^c A^{\mu d} A^{\nu f}$	$Z_3^2 Z_g^2$	$\equiv Z_4$
$+ \partial_\mu \chi_a^* \partial^\mu \chi_a$	$\tilde{Z}_3$	
$- g \partial^\mu \chi_a^* f_{abc} A_\mu^b \chi_c$	$Z_g \tilde{Z}_3 \sqrt{Z_3}$	$\equiv \tilde{Z}_1$
$-\frac{1}{2a} (\partial \cdot A)^2$	$Z_3/Z_3 = 1$	
$+ i \bar{\psi} \not{\partial} \psi$	Z_2	
$- m \bar{\psi} \psi$	$Z_2 Z_m$	
$- g \bar{\psi} \not{A} \psi$	$Z_g Z_2 \sqrt{Z_3}$	$\equiv Z_{1F}$

This leads to renormalized propagators and vertices:

 $$= \frac{i}{Z_2 \not{p} - Z_2 Z_m m_R}$$

 $$= -i\, Z_g\, Z_2 \sqrt{Z_3}\; g_R\, t^a \gamma^\mu$$

 $$= \frac{\delta_{ab}\; i\, d_{\mu\nu}(k)}{Z_3\, k^2} \qquad \text{etc.}$$

Note: the precise form of Zs depends upon the order of perturbative series up to which we calulate.

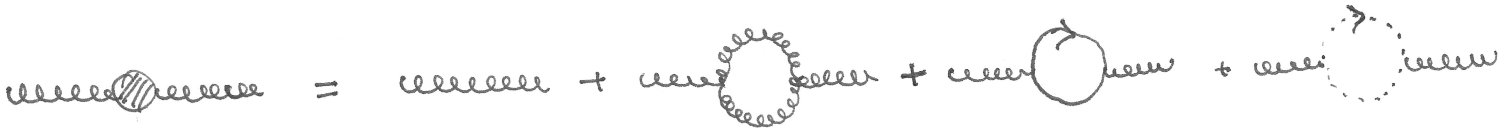
To LO $Z_i = 1$ for all i and the bare quantities are equal to the renormalized ones.

- The Zs are defined to absorb all UV divergences that appear to a given order in perturbation theory.

- There is a freedom in defining the finite part of the Zs, which leads to an arbitrariness in the fixed order finite result of a calculation. This arbitrariness is refered to as "Renormalization Scheme Dependence".

- Moreover, dimensional regularization makes the coupling constant (bare and renormalized) dimensionful! To get a dimensionless coupling constant one defines $g_0 = g\mu^\varepsilon$ or $g_R^D = g_R \mu^\varepsilon$. The scale μ is arbitrary and introduces a scale uncertainty to every fixed order calculation (including Leading Order ones).

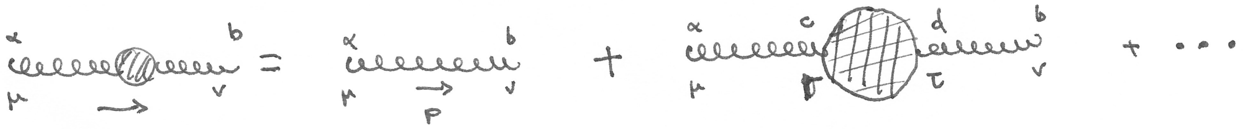
RENORMALIZING THE GLUON PROPAGATOR.



note: we will assume that the gluon propagator is used in an NLO calculation, so we will ignore any higher order term in α_s.

In particular, all the Zs that appear in the three loop diagrams will be discarded since they would only contribute at NNLO.

note: We will stick to the feynman gauge $a = \frac{1}{\lambda} = 1$, so the gluon propagator is $\delta_{ab} \dfrac{-i g_{\mu\nu}}{p^2}$



$$= \frac{\delta_{ab}(-i g_{\mu\nu})}{Z_3 p^2} + \frac{\delta_{ac}(-i g_{\mu\sigma})}{Z_3 p^2} \Pi^{\sigma\tau}_{cd}(p)\, \delta_{db} \frac{-i g_{\tau\nu}}{Z_3 p^2} + \dots$$

Assuming $\Pi^{\sigma\tau}_{cd}(p) = \delta_{cd} \left(-g^{\sigma\tau} + \frac{p^\sigma p^\tau}{p^2}\right) \Pi(p^2)$ (transversality preserved)

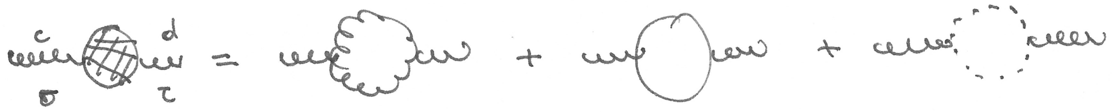
we have

$$= \delta_{ab} \frac{-i g_{\mu\nu}}{Z_3 p^2} + \delta_{ab} (-i)(g_{\mu\nu})(-i) \frac{1}{Z_3 p^2} \Pi(p^2) \frac{1}{Z_3 p^2}$$

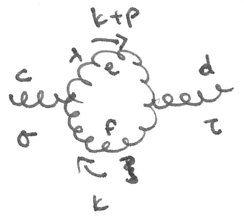
$$= \frac{\delta_{ab}(-i g_{\mu\nu})}{Z_3 p^2 + i\,\Pi(p^2)}$$

$\Pi(p^2)$ is expected to be divergent: it contains the three bubble diagrams above.

Z_3 will be set to $Z_3 = 1 + \delta Z_3$ with δZ_3 cancelling the divergent piece of $\Pi(p^2)$.



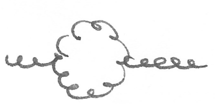
$$= \delta_{cd} \left(-g_{\sigma\tau} + \frac{p_\sigma p_\tau}{p^2} \right) \Pi(p^2)$$



$$= \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} (-ig)\, f^{fce} f^{edf} (-ig) \frac{V_{\sigma\lambda\xi}(p,-p-k,k)\, V_{\lambda\tau\xi}(k+p,-p,-k)}{k^2 (k+p)^2}$$

$$= \frac{1}{2} g^2 N_c \delta_{cd} \int \frac{d^D k}{(2\pi)^D} \frac{A_{\sigma\tau}(k,p)}{k^2 (k+p)^2}$$

with $A_{\sigma\tau} = g_{\sigma\tau}(2k^2 + 5p^2 + 2p\cdot k) + k_\mu k_\nu (4D-6)$
$+ (p_\mu k_\nu + k_\mu p_\nu)(2D-3) + p_\mu p_\nu (D-6)$. (eq. *)



$$= \dots = \frac{1}{2} \frac{g^2 N_c \delta_{cd}}{16\pi^2} (4\pi)^\epsilon (-p^2)^{-\epsilon} \Gamma(\epsilon) \frac{B(2-\epsilon, 2-\epsilon)}{1-\epsilon} \cdot$$

$$\cdot \left[(19-12\epsilon) p^2 g_{\sigma\tau} - 2(11-7\epsilon) p_\sigma p_\tau \right]$$

(eq. **)

$\boxed{D = 4 - 2\epsilon}$

note: the gluon loop correction is <u>not</u> transverse!

note: step-by-step guide to go from (eq. *) to (eq. **) at pages 10-13.

$$= (-1) \int \frac{d^D k}{(2\pi)^D} \frac{(-ig)^2 f^{ecf} f^{fde} (-i)^2 (p+k)_\mu k_\nu}{k^2 (p+k)^2}$$

$$= \cdots$$

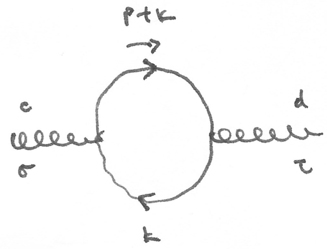
$$= \frac{1}{2} \frac{g^2 N_c \delta_{cd}}{16\pi^2} (4\pi)^\varepsilon (-p^2)^{-\varepsilon} \Gamma(\varepsilon) \frac{B(2-\varepsilon, 2-\varepsilon)}{1-\varepsilon} \cdot$$

$$\cdot \left[p^2 g_{\sigma\tau} + 2(1-\varepsilon) p_\sigma p_\tau \right]$$

So  $= \frac{1}{2} \frac{g^2 N_c \delta_{cd}}{16\pi^2} (4\pi)^\varepsilon (-p^2)^{-\varepsilon} \Gamma(\varepsilon) \frac{B(2-\varepsilon,2-\varepsilon)}{1-\varepsilon}$

$$\Big[\underbrace{(19 - 12\varepsilon + 1)}_{= 20 - 12\varepsilon} p^2 g_{\sigma\tau} - \underbrace{(22 - 14\varepsilon - 2 + 2\varepsilon)}_{= 20 - 12\varepsilon} p_\sigma p_\tau \Big]$$

The sum is transverse!



$$= (-1) n_f \int \frac{d^D k}{(2\pi)^D} (-ig)^2 i^2 \, \mathrm{Tr}(t^c t^d) \frac{\mathrm{Tr}\left[\gamma^\tau \not{k} \gamma^\sigma (\not{k} + \not{p})\right]}{k^2 (p+k)^2}$$

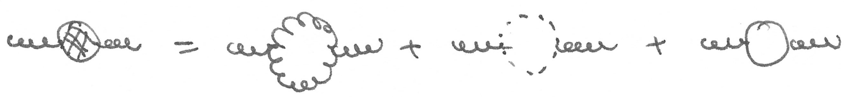
$$= \cdots$$

$$= -4 \frac{g^2}{16\pi^2} \frac{\delta_{ab}}{2} n_f \Gamma(\varepsilon) (4\pi)^\varepsilon \left(p^2 g_{\sigma\tau} - p_\sigma p_\tau\right) \cdot$$

$$\cdot \int_0^1 dx \, x(1-x) \left[m^2 - x(1-x) p^2\right]^{-\varepsilon}$$

note: when $m=0 \to \int_0^1 dx \, x(1-x) \, x^{-\varepsilon} (1-x)^{-\varepsilon} (-p^2)^{-\varepsilon}$

$$= (-p^2)^{-\varepsilon} \int_0^1 dx \, x^{1-\varepsilon} (1-x)^{1-\varepsilon} = (-p^2)^{-\varepsilon} B(2-\varepsilon, 2-\varepsilon).$$

So 

$$= \frac{g^2}{16\pi^2} (4\pi)^{\epsilon} (-p^2)^{-\epsilon} \, \Gamma(\epsilon) \, B(2-\epsilon, 2-\epsilon) \, \cdot$$

$$\cdot \Big[\qquad\qquad\qquad \Big]$$

$$= -\frac{g_0^2}{16\pi^2} N_c \left[\frac{10}{6} \left(\frac{1}{\epsilon} - \gamma - \log \frac{-p^2}{4\pi\mu^2} \right) + \frac{31}{9} \right]$$

$$+ \frac{g_0^2}{16\pi^2} \frac{n_F}{2} \frac{4}{3} \left[\left(\frac{1}{\epsilon} - \gamma - \log \frac{-p^2}{\mu^2} + \log 4\pi \right) - 6 \int_0^1 dx \, x(1-x) \, \cancel{\log x(1-x)} \, \log\left[x(1-x) - \frac{m^2}{p^2} \right] \right]$$

where we have introduced $\boxed{g = g_0 \mu^{\epsilon}}$

What exactly is Z_3? The ren. propagator is $\dfrac{\delta_{ab}(-i g_{\mu\nu})}{Z_3 p^2 + \Pi(p^2)}$

This depends on the scheme:

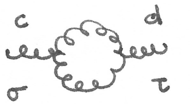
MS scheme : $Z_3 = 1 - \dfrac{\alpha_s}{4\pi} \left[\dfrac{4}{3} \dfrac{n_F}{2} - \dfrac{1}{2} N_c \dfrac{10}{3} \right] \dfrac{1}{\epsilon}$

Zs only cancel poles

$\overline{MS}$ scheme : $Z_3 = 1 - \dfrac{\alpha_s}{4\pi} \left[\dfrac{4}{3} \dfrac{n_F}{2} - \dfrac{1}{2} N_c \dfrac{10}{3} \right] \left(\dfrac{1}{\epsilon} - \gamma + \log 4\pi \right)$

<u>note</u>: in those schemes Z_3 is μ-independent.

EXPLICIT GLUON BUBBLE CALCULATION ①



$$= \frac{g^2}{2} N_c \delta_{cd} \int \frac{d^D k}{(2\pi)^D} \frac{A_{\sigma\tau}(k,p)}{k^2 \, (k+p)^2}$$

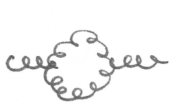
COMMENT

The form of $A_{\sigma\tau}$ is the result of lengthy but straightforward algebra

with $A_{\sigma\tau} = g_{\sigma\tau}\left(2k^2 + 5p^2 + 2p\cdot k\right) + k_\sigma k_\tau (4D-6)$

$$+ \left(p_\sigma k_\tau + k_\sigma p_\tau\right)(2D-3) + p_\sigma p_\tau (D-6)$$

Introducing feynman parameters



$$= \frac{g^2}{2} N_c \delta_{cd} \int dx \int \frac{d^D k}{(2\pi)^D} \frac{A_{\sigma\tau}}{\left[(1-x)k^2 + x(k+p)^2\right]^2}$$

The denominator is $(1-x)k^2 + x(k+p)^2 = k^2 + 2k\,x\cdot p + x p^2$

Completing the square

$$= (k+xp)^2 - x^2p^2 + xp^2 = (k+xp)^2 - \Delta$$

with $\Delta = -p^2 x(1-x)$

$$= \frac{g^2}{2} N_c \delta_{cd} \int dx \int \frac{d^D k}{(2\pi)^D} \frac{A_{\sigma\tau}}{\left[(k+xp)^2 - \Delta\right]^2}$$

Shift the momentum integral $k^\mu = \ell^\mu - x p^\mu$

Shifting the loop momentum

Then the numerator becomes

$$A_{\sigma\tau} = g_{\sigma\tau}\left(2(\ell - xp)^2 + 5p^2 + 2p\cdot(\ell - xp)\right) + \left(\ell^\sigma - xp^\sigma\right)\left(\ell^\tau - xp^\tau\right)(4D-6)$$

$$+ \left[p^\sigma\left(\ell^\tau - xp^\tau\right) + p^\tau\left(\ell^\sigma - xp^\sigma\right)\right](2D-3) + p^\sigma p^\tau (D-6)$$

note: terms linear in ℓ lead to $\int_{-\infty}^{+\infty} \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu}{(\ell^2-\Delta)^2} = 0$ so we'll drop them from $A_{\sigma\tau}$.

$$A_{\sigma\tau} = g_{\sigma\tau}\left[2\ell^2 + 2x^2p^2 + 5p^2 - 2xp^2\right] + \ell^\sigma \ell^\tau (4D-6) + x^2 p^\sigma p^\tau (4D-6)$$

$$+ \left[-xp^\sigma p^\tau \cdot 2\right](2D-3) + p^\sigma p^\tau (D-6)$$

Dropping linear terms in ℓ

$\boxed{\text{EXPLICIT GLUON BUBBLE CALCULATION}}$ ②

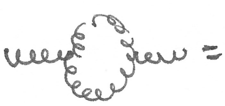
$$A_{\sigma\tau} = g_{\sigma\tau}\left[2l^2 + 2x(x-1)p^2 + 5p^2\right] + l_\sigma l_\tau (4D-6) + p_\sigma p_\tau\left[x^2(4D-6) - x(4D-6) + D - 6\right]$$

$$= g_{\sigma\tau}\left[2l^2 + 2x(x-1)p^2 + 5p^2\right] + l_\sigma l_\tau (4D-6) + p_\sigma p_\tau\left[x(x-1)(4D-6) + D - 6\right]$$

note: $x(x-1)p^2 \equiv \Delta$, so

$$A_{\sigma\tau} = g_{\sigma\tau}\left[2l^2 + 2\Delta + 5p^2\right] + l_\sigma l_\tau (4D-6) + \frac{p_\sigma p_\tau}{p^2}\left[\Delta(4D-6) + p^2(D-6)\right].$$

which means

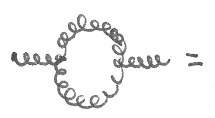


$$= \frac{g^2}{2} N_c \delta_{cd} \int dx \int \frac{d^Dl}{(2\pi)^D} \frac{g_{\sigma\tau}(2l^2 + \Delta 2 + 5p^2) + l_\sigma l_\tau (4D-6) + \frac{p_\sigma p_\tau}{p^2}\left[\Delta(4D-6) + p^2(D-6)\right]}{(l^2-\Delta)^2}$$

$\boxed{\text{Rewritting } l_\sigma l_\tau \text{ in terms of } l^2}$

note: The second term in the integral is $\displaystyle\int \frac{d^Dl}{(2\pi)^D} \frac{l_\sigma l_\tau (4D-6)}{(l^2-\Delta)^2} = (4D-6)\int \frac{d^Dl}{(2\pi)^D} \frac{g_{\sigma\tau}\, l^2/D}{(l^2-\Delta)^2}$

because $\displaystyle l_\mu l_\nu = \frac{l^2}{D} g_{\mu\nu}$ + terms that are odd and vanish in the integral.

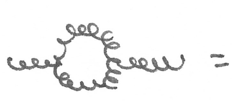


$$= \frac{g^2}{2} N_c \delta_{cd} \int dx \int \frac{d^Dl}{(2\pi)^D} \frac{g_{\sigma\tau}\left[\left(2 + \frac{4D-6}{D}\right) l^2 + 2\Delta + 5p^2\right] + M_{\sigma\tau}}{(l^2-\Delta)^2}$$

$$M_{\sigma\tau} = \frac{p_\sigma p_\tau}{p^2}\left[\Delta(4D-6) + p^2(D-6)\right]$$

$\boxed{\text{Perform the Wick rotation}}$

Now perform the Wick rotation: $l^0 \to i l^0_E$, $l^2 \to -l_E^2$



$$= \frac{g^2}{2} N_c \delta_{cd} \int dx \int \frac{i\, d^Dl_E}{(2\pi)^D} \frac{-g_{\sigma\tau}\left(2 + \frac{4D-6}{D}\right) l_E^2 + g_{\sigma\tau}(2\Delta + 5p^2) + M_{\sigma\tau}}{(l_E^2+\Delta)^2}$$

$\boxed{\text{Perform the Loop integral}}$

note: $$\int \frac{d^Dl_E}{(2\pi)^D} \frac{l_E^2}{(l_E^2+\Delta)^2} = \frac{D/2 \cdot \Delta}{\varepsilon - 1} \Delta^{-\varepsilon}(4\pi)^\varepsilon \Gamma(\varepsilon)$$

$$\int \frac{d^Dl_E}{(2\pi)^D} \frac{1}{(l_E^2+\Delta)^2} = \Delta^{-\varepsilon}(4\pi)^\varepsilon \Gamma(\varepsilon)$$

$\Big\}$ see <u>SUPPLEMENT</u> for a derivation of those.

Performing the l_E integral we get


$$= \frac{ig^2}{2} N_c \delta_{cd} \cdot \int dx \left[-g_{\sigma\tau}\left(2+\frac{4D-6}{D}\right)\frac{D}{2}\frac{\Delta}{\varepsilon-1} + g_{\sigma\tau}(2\Delta + 5p^2) + M_{\sigma\tau}\right] \frac{\Delta^{-\varepsilon}\Gamma(\varepsilon)}{(4\pi)^{\varepsilon}}$$

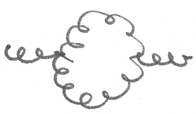
note: $\left(2+\frac{4D-6}{D}\right)\frac{D}{2} = D + 2D-3 = 3(D-1) = 9-6\varepsilon$


$$= \frac{ig^2}{2} N_c \delta_{cd} \Big\{ +g_{\sigma\tau}\frac{(9-6\varepsilon)}{1-\varepsilon}\int dx\, \Delta^{1-\varepsilon} + g_{\sigma\tau}\, 2\int dx\, \Delta^{1-\varepsilon} + g_{\sigma\tau} 5p^2 \int \Delta^{-\varepsilon}$$
$$+ \frac{p_\sigma p_\tau}{p^2}(4D-6)\int dx\, \Delta^{1-\varepsilon} + \frac{p_\sigma p_\tau}{p^2}\cdot p^2 (D-6)\int dx\, \Delta^{-\varepsilon}\Big\} \Gamma(\varepsilon)(4\pi)^{\varepsilon}$$

$\boxed{\text{Performing the Feyn. param. integral}}$

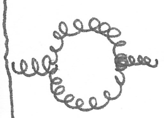
note: $\int dx\, \Delta^{1-\varepsilon} = \int dx\, (-p^2)^{1-\varepsilon} x^{1-\varepsilon}(1-x)^{1-\varepsilon} = (-p^2)(-p^2)^{-\varepsilon}\, B(2-\varepsilon, 2-\varepsilon)$

$$\int dx\, \Delta^{-\varepsilon} = (-p^2)^{-\varepsilon}\int dx\, x^{-\varepsilon}(1-x)^{-\varepsilon} = (-p^2)^{-\varepsilon}\frac{\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)}$$
$$= (-p^2)^{-\varepsilon}\frac{(3-2\varepsilon)(2-2\varepsilon)}{(1-\varepsilon)(1-\varepsilon)}\frac{\Gamma(2-\varepsilon)\Gamma(2-\varepsilon)}{\Gamma(4-2\varepsilon)}$$
$$= (-p^2)^{-\varepsilon}\frac{(3-2\varepsilon)\cdot 2}{1-\varepsilon} B(2-\varepsilon, 2-\varepsilon).$$


$$= \frac{ig^2}{2} N_c \delta_{cd} \Big[g_{\sigma\tau}\frac{3(3-2\varepsilon)}{1-\varepsilon}(-p^2) + g_{\sigma\tau}\cdot 2(-p^2) + g_{\sigma\tau} 5p^2 \frac{2(3-2\varepsilon)}{1-\varepsilon}$$
$$+ \frac{p_\sigma p_\tau}{p^2}(4D-6)(-p^2) + \frac{p_\sigma p_\tau}{p^2} p^2 (D-6)\frac{2(3-2\varepsilon)}{1-\varepsilon}\Big\} \Gamma(\varepsilon)(4\pi)^{\varepsilon}(-p^2)^{-\varepsilon} B(2-\varepsilon, 2-\varepsilon)$$

$\boxed{\text{collecting terms}}$

$$= \frac{ig^2}{2} N_c \delta_{cd} p^2 \Big[g_{\sigma\tau}\left(-9+6\varepsilon - 2(1-\varepsilon) + 30 - 20\varepsilon\right)$$
$$+ \frac{p_\sigma p_\tau}{p^2}\Big[(-10+8\varepsilon)(1-\varepsilon) + (6-4\varepsilon)(-2-2\varepsilon)\Big]\Big] \Gamma(\varepsilon)(4\pi)^{\varepsilon}(-p^2)^{-\varepsilon} \times \frac{B(2-\varepsilon, 2-\varepsilon)}{1-\varepsilon}$$


$$\boxed{= \frac{ig^2}{2} N_c \delta_{cd} p^2 \left[g_{\sigma\tau}\left(19-12\varepsilon\right) + \frac{p_\sigma p_\tau}{p^2}(-22+14\varepsilon)\right] \Gamma(\varepsilon)(4\pi)^{\varepsilon}(-p^2)^{-\varepsilon}\frac{B(2-\varepsilon,2-\varepsilon)}{1-\varepsilon}}$$

EXPLICIT GLUON BUBBLE CALCULATION

SUPPLEMENT

$$I_{p,q} = \int \frac{d^D l_E}{(2\pi)^D} \frac{(l_E^2)^p}{(l_E^2+\Delta)^q} = \int_0^\infty dx\, x^{D-1} \frac{d\Omega_D}{(2\pi)^D} \frac{(x^2)^p}{(x^2+\Delta)^q} \qquad (x = |l_E|)$$

Set $x^2 = \Delta y^2 \Rightarrow x\,dx = y\,dy\,\Delta$

$$I_{p,q} = \int_0^\infty y\,dy \frac{\Delta (\Delta y^2)^{\frac{D-2}{2}} (\Delta y^2)^p}{\Delta^q (y^2+1)^q} \cdot \underbrace{\int \frac{d\Omega_D}{(2\pi)^D}}$$

the D-dimensional angular measure

$$= \frac{2\pi^{D/2}}{\Gamma(D/2)(2\pi)^D} = \frac{2}{(4\pi)^{D/2}\Gamma(D/2)} \equiv C_D$$

$$I_{p,q} = \int_0^\infty \frac{dy^2}{2} \Delta^{D/2+p-q} \frac{(y^2)^{D/2+p-1}}{(1+y^2)^{D/2+p+q-p-D/2}} \cdot C_D$$

$$I_{p,q} = \int_0^\infty \frac{du}{2} \Delta^{D/2+p-q} \frac{u^{r-1}}{(1+u)^{r+s}} \cdot C_D \quad \text{with} \quad r = D/2+p \quad s = q-p-D/2$$

Now $\displaystyle\int_0^\infty du \frac{u^{r-1}}{(1+u)^{r+s}} = B(r,s) \equiv \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$, so

$$I_{p,q} = C_D \cdot \frac{1}{2} \Delta^{D/2+p-q} \frac{\Gamma(D/2+p)\,\Gamma(q-p-D/2)}{\Gamma(q)}$$

examples

$\boxed{p=0 \quad q=2}$

$$\int \frac{d^D l_E}{(2\pi)^D} \frac{(l_E^2)^0}{(l_E^2+\Delta)^2} = \int \frac{d^D l_E}{(2\pi)^D} \frac{1}{(l_E^2+\Delta)^q}$$

$$= \frac{2}{(4\pi)^{D/2}\Gamma(D/2)} \frac{\Delta^{D/2-2}}{2} \frac{\Gamma(D/2)\,\Gamma(2-D/2)}{\Gamma(2)}$$

$$= \frac{\Delta^{-\varepsilon}\,\Gamma(\varepsilon)\,(4\pi)^\varepsilon}{16\pi^2}$$

$\boxed{p=1 \quad q=2}$

$$\int \frac{d^D l_E}{(2\pi)^D} \frac{l_E^2}{(l_E^2+\Delta)^2} = \frac{\Delta^{1-\varepsilon}(4\pi)^\varepsilon}{16\pi^2} D/2 \frac{\Gamma(\varepsilon)}{(\varepsilon-1)}$$