Classical equations of motion and gauge fixing

Summary of classical QCD Lagrangian (with one flavour)

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \bar{\psi}(i\not{D} - m)\psi$$

Covariant derivative $D_\mu = \partial_\mu + ig A^a_\mu t^a$ ensures that $\mathcal{L}$ is invariant under gauge transformations (in infinitesimal form)

$$\begin{cases} \psi(x) \to \psi(x) - i\theta^a(x) t^a \psi(x) \\ A^a_\mu(x) \to A^a_\mu(x) + \frac{1}{g} D^{ab}_\mu \theta^b(x) \end{cases}$$

Constructing the action $S[A, \psi, \bar{\psi}] = \int d^4x\, \mathcal{L}(x)$ and imposing $\delta S = 0$ under variations of the fields that vanish at the boundary we obtain the classical equations of motion (Euler-Lagrange equations)

$$\begin{cases} \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \quad \Rightarrow \quad D^{ab}_\mu F^{\mu\nu}_b = g\,\bar{\psi}\, t^a \gamma_\nu \psi \\ \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 \quad \Rightarrow \quad (i\not{D} - m)\psi = 0 \end{cases}$$

Note that unlike electromagnetism the equations of motion for the field strength $F_{\mu\nu}$ involve the gluon field A_μ

Classical equations of motion for gauge field are not invertible. This is a consequence of gauge invariance. To invert the equations of motion one eliminates redundant configurations by imposing a "gauge fixing" condition ~~which fixes~~

$$G_F[A^\theta] = 0 \quad \Rightarrow \quad \text{fix the value of } \theta^a(x)$$

Problems with quantisation of the QCD Lagrangian

- Canonical quantisation

Consider the pure Yang-Mills classical Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$$

Computing the momentum conjugate to A^a_μ gives

$$\pi^a_\mu = \frac{\partial \mathcal{L}}{\partial(\partial^0 A^\mu)} = -F^a_{0\mu} \Rightarrow \pi^a_0 = 0$$

This implies that one cannot have Lorentz covariant commutation rules.
One then adds to the Lagrangian a gauge-fixing term

$$\mathcal{L}_{GF} = -\frac{\lambda}{2}\left(G_F[A]\right)^2$$

such that $G_F[A^\theta] = 0$ determines the value of $\theta^a(x)$. We then have

$$\pi^a_\mu = -F^a_{0\mu} - \lambda\, G_F[A] \frac{\partial G_F[A]}{\partial(\partial^0 A^\mu)}$$

For instance in the "Lorentz gauge" $\partial^\mu_\cdot A^a_\mu = 0$ one has

$$\pi^a_\mu = -F^a_{0\mu} - \lambda\, g_{0\mu}(\partial^\nu A^a_\nu) \Rightarrow \pi^a_0 = -\lambda(\partial^\nu A^a_\nu)$$

so that the "gauge fixing" condition $\partial^\mu A^a_\mu = 0$ is not consistent with Lorentz covariant canonical commutation rules.

Releasing the condition $\partial^\mu A^a_\mu = 0$ and proceeding with canonical quantisation gives a Hilbert space with indefinite metric.
One needs to impose non-trivial conditions on a subspace of the Hilbert space such that states in this subspace have positive norm.
These are the physical states.

• Path-integral quantisation

Constructing a "positive norm" Hilbert space for physical states is nontrivial. The same problem occurs in "path-integral" formulation. There one constructs a generating functional

$$Z[J] = \int [dA]\, e^{i\int d^4x \left[\mathcal{L}_{YM}(x) + J^\mu_a(x) A^a_\mu(x)\right]}$$

from which one can construct time-ordered products of operators a.k.a. "Green function" $G_{\mu_1 \ldots \mu_n}(x_1 \ldots x_n)$

$$\langle 0| T[A_{\mu_1}(x_1) \ldots A_{\mu_n}(x_n)]|0\rangle = \frac{\int (dA)\, A_{\mu_1}(x_1) \ldots A_{\mu_n}(x_n)\, e^{iS_{YM}[A]+ \ldots}}{\int (dA)\, e^{iS_{YM}(A) + \ldots}} =$$

$$= \frac{1}{Z[0]} \left(\frac{-i\delta}{\delta J_{\mu_1}(x_1)}\right) \ldots \left(\frac{-i\delta}{\delta J_{\mu_n}(x_n)}\right) Z[J]\Big|_{J=0}$$

However, all configurations $A^\theta_\mu(x)$ connected via a gauge transformation give the same value for the integral. One can however fix A^θ corresponding to a "gauge-fixing" condition $G_F[A^\theta] = 0$ and integrate over all θ' such that $S_{YM}[A^{\theta\theta'}] = S_{YM}[A^\theta]$. This gives the correct functional integral $Z(0)$

$$Z[0] = \int (dA) \left(\det\left[\frac{\delta G_F[A^\theta]}{\delta\theta}\right]\right) e^{i\int d^4x \left(\mathcal{L}_{YM}(x) - \frac{1}{2}(G_F[A])^2\right)} =$$

$$= \int (dA)(d\chi)(d\chi^*)\, e^{i\int d^4x \left[\mathcal{L}_{YM}(x) + \mathcal{L}_{GF}(x) + \mathcal{L}_{FP}(x)\right]}$$

Where $\mathcal{L}_{FP}$ is the Lagrangian for the "Fadeev-Popov ghosts"

$$\mathcal{L}_{FP}(x) = -\int d^4y\, \chi^*_a(x) \frac{\delta G_F[A^\theta_a(x)]}{\delta\theta_b(y)} \chi_b(y)$$

The field $\chi_a(x)$ is called "ghost", it is a <u>scalar, anticommuting</u> field

• Gauge fixing and ghost Lagrangean

The ghost Lagrangean depends of course on the gauge fixing condition $G_F(A)$ and has to be evaluated at the value of the field A_μ that satisfies the condition $G_F(A] = 0$

We basically distinguish two kinds of gauge-fixing conditions

1) Physical gauges $n^\mu A_\mu^a = 0$ n^μ fixed vector

$$G_F[A^\theta] = n^\mu A_\mu^{a\theta} = n^\mu \left(A_\mu^a + \frac{1}{g} D_\mu^{ab} \theta_b\right)$$

$$\frac{\delta G_F(A_a^\theta(x)]}{\delta \theta_b(y)} = n^\mu D_\mu^{ab} \delta^4(x-y)$$ factor $\frac{1}{g}$ omitted, just normalisation of ghost fields.

$$\mathcal{L}_{FP} = -\chi_a^*(x) n^\mu D_\mu^{ab} \chi_b(x) = -\chi_a^*(x) n^\mu \partial_\mu \chi_a(x)$$

where we have used the gauge-fixing condition $n^\mu A_\mu = 0$. We note that the ghosts do not interact with the gluons

2) Covariant gauge $\partial^\mu A_\mu^a = 0$

$$G_F(A^\theta) = \partial^\mu A_\mu^{a\theta} = \partial^\mu \left(A_\mu^a + \frac{1}{g} D_\mu^{ab} \theta^b\right)$$

$$\frac{\delta G_F(A_a^\theta(x))}{\delta \theta_b(y)} = \partial^\mu D_\mu^{ab} \delta^4(x-y)$$

$$\mathcal{L}_{FP} = -\chi_a^* \partial^\mu D_\mu^{ab} \chi_b = \partial^\mu \chi_a^* D_\mu^{ab} \chi_b + \text{surface term} =$$

$$= \partial^\mu \chi_a^* \partial_\mu \chi_a - \underbrace{g\, \partial^\mu \chi_a^* f_{abc} A_\mu^b \chi_c}_{\text{interaction term of ghosts with gluons}}$$

Free gluon propagator

We split the Yang-Mills Lagrangean in a "free" part and an "interaction" part $\mathcal{L}_{YM} = \mathcal{L}_0 + \mathcal{L}_I$, where

$$\mathcal{L}_0 = -\frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}\right) - \frac{\lambda}{2}\left(G_F[A]\right)^2$$

After integration by parts, one can write

$$\mathcal{L}_0 = \frac{1}{2} A_\mu^a \left(D^{\mu\nu}\right)^{-1}_{ab} A_\nu^b$$

where the operator $(D^{\mu\nu})^{-1}_{ab}$ depends of course on the gauge-fixing condition, and is invertible. Its inverse $D^{ab}_{\mu\nu}$ is the "free" gluon "propagator" and is the basic building block of perturbative expansion of Green functions.

The generating functional $Z[J]$, can be expanded in powers of the coupling as follows:

$$Z[J] = \int [dA]\, \Delta_F(A)\, e^{i\left(A D^{-1} A + S_I[A] + JA\right)} = \qquad \left[\Delta_F(A) \equiv \det\left[\frac{\delta G_F[A^\theta]}{\delta\theta}\right]\right]$$

$$= \sum_{n=0}^{+\infty} \frac{(+i)^n}{n!} \int [dA]\, \Delta_F(A)\, (S_I[A])^n \cdot e^{i\left(\frac{1}{2} A D^{-1} A + JA\right)} =$$

$$= \sum_{n=0}^{+\infty} \frac{i^n}{n!}\, \Delta_F\left[-\frac{i\delta}{\delta J}\right] \left(S_I\left[\frac{-i\delta}{\delta J}\right]\right)^n Z_0[J]$$

where

$$Z_0[J] = \int [dA]\, e^{i\left(\frac{1}{2} A D^{-1} A + JA\right)} \propto e^{-\frac{i}{2}\int d^4x\, d^4y\, J_a^\mu(x)\, D^{ab}_{\mu\nu}(x-y)\, J_b^\nu(y)}$$

so that each time we act with two functional derivatives we bring down a free gluon propagator $i D^{ab}_{\mu\nu}(x-y)$

One then defines a "Feynman" propagator $[D^{ab}_{\mu\nu}]_F(x-y) = i\, D^{ab}_{\mu\nu}(x-y)$ satisfying

$$(D^{-1}_{\mu\rho})^{ac}\,[D^{\rho\nu}_{cb}(x)]_F = i\,\delta^a_b\, g_\mu{}^\nu\, \delta^4(x)$$

1) Physical / transverse gauge $n^\mu A_\mu^a = 0$

$$\mathcal{L}_0 = \frac{1}{2} A_\mu^a \left[\Box g^{\mu\nu} - (\partial^\mu\partial^\nu + \lambda n^\mu n^\nu)\right] \delta^{ab} A^\nu_b$$

The equation is easily inverted via a Fourier transform

$$[D^{ab}_{\mu\nu}(x)]_F = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x}\, \frac{i}{k^2+i\epsilon}\, d_{\mu\nu}(k)\, \delta^{ab} \equiv \tilde{D}^{ab}_{\mu\nu}(k)$$

$$d_{\mu\nu}(k) = -g_{\mu\nu} + \frac{n_\mu k_\nu + k_\mu n_\nu}{n\cdot k} - \left(n^2 + \frac{k^2}{\lambda}\right)\frac{k_\mu k_\nu}{(n\cdot k)^2}$$

2) Covariant / Lorentz gauge $\partial^\mu A_\mu^a = 0$

$$\mathcal{L}_0 = \frac{1}{2} A_\mu^a \left(\Box g^{\mu\nu} - (1-\lambda)\partial^\mu\partial^\nu\right) \delta^{ab} A^\nu_b$$

In this case $\tilde{D}^{ab}_{\mu\nu}(k) = \delta^{ab}\, \dfrac{i}{k^2+i\epsilon}\, d_{\mu\nu}(k)$

where $d_{\mu\nu}(k) = -g_{\mu\nu} + \left(1-\dfrac{1}{\lambda}\right)\dfrac{k_\mu k_\nu}{k^2+i\epsilon}$

Two remarks

a) Sometimese $1/\lambda$ is referred to as α [see Muta]

b) The prescription $k^2 \to k^2+i\epsilon$ is such that $[D^{ab}_{\mu\nu}(x-y)]_F = {}_I\langle 0|T[(A^a_\mu)_I(x)(A^b_\nu)_I(y)]|0\rangle_I$
when $(A^a_\mu)_I(x)$ is a "free" field such that $(D^{ab}_{\mu\nu})^{-1}(A^\nu_b)_I(x) = 0$

Perturbative expansion of Green functions

We are interested in Green's functions such as

$$G^{2n,m}_{\mu_1 - \mu_m}(x_1, y_1, - x_n, y_n, z_1 - z_m) = \langle 0 | T[\psi(x_1)\bar\psi(y_1) - \psi(x_n)\bar\psi(y_n) A_{\mu_1}(z_1) - A_{\mu_m}(z_m)] | 0\rangle$$

and in its Fourier transform

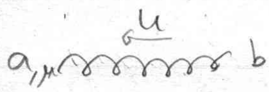
$$\tilde G^{2n,m}_{\mu_1 - \mu_m}(p_1, \bar p_1 - p_n, \bar p_m, k_1 - k_m)(2\pi)^4 \delta^4\left(\sum_i p_i + \sum_i \bar p_i + \sum_i k_i\right) :=$$

$$\equiv \int \prod_{i=1}^n d^4x_i\, e^{+ip_i x_i} \prod_{i=1}^n d^4y_i\, e^{i\bar p_i y_i} \prod_{j=1}^m d^4 z_i\, e^{ik_i z_i}\, G_{\mu_1 - \mu_m}(x_1, y_1, --- z_m)$$

The momentum conservation δ is due to the fact that G is invariant under translation. The Green function $\tilde G$ is constructed by fixing the proper number of external legs, n quarks, n antiquarks and m gluons, and connecting them with free propagators and interaction vertices. We now consider a covariant gauge

- Propagators

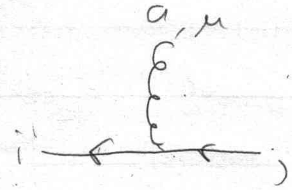
$$\delta_{ij} \; i\frac{\not p + m}{p^2 - m^2 + i\varepsilon} \Leftarrow \bar\psi(i\not\partial - m)\psi \qquad \text{quark}$$



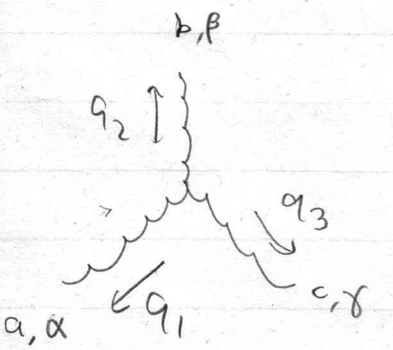
$$\delta_{ab} \frac{i}{k^2 + i\varepsilon} d_{\mu\nu}(k) \Leftarrow \frac{1}{2} A^a_\mu (D^{\mu\nu}_{ab})^{-1} A^b_\nu \qquad \text{gluon}$$

$$\delta_{ab} \frac{i}{k^2 + i\varepsilon} \Leftarrow \partial_\mu \chi^*_a \partial^\mu \chi_a \qquad \text{ghost}$$

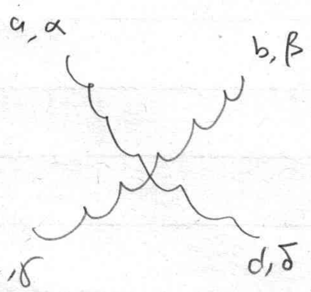
- Interaction vertices



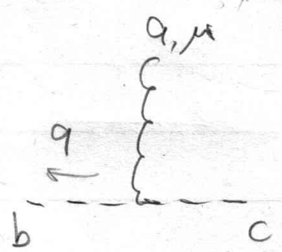
$$-ig\gamma^\mu t^a_{ij} \Leftarrow -g\bar{\Psi}_i\gamma^\mu A^a_\mu t^a_{ij}\Psi_j$$



$$-ig(if^{abc}) \times$$
$$\times [g_{\alpha\beta}(q_1-q_2)_\gamma + g_{\beta\gamma}(q_2-q_3)_\alpha + g_{\gamma\alpha}(q_3-q_1)_\beta]$$
$$\Uparrow$$
$$\frac{g}{2} f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) A^\mu_b A^\nu_c$$



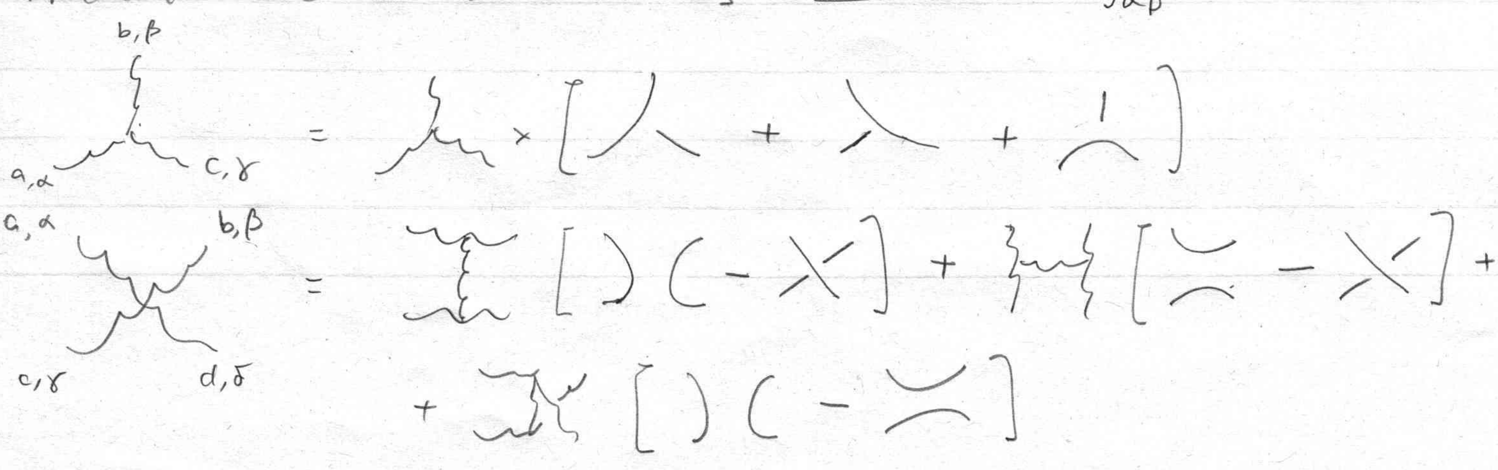
$$-ig^2 [f^{abe}f^{cde}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) +$$
$$+ f^{ace}f^{bde}(g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\delta}g_{\beta\gamma}) +$$
$$+ f^{ade}f^{cbe}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\beta}g_{\gamma\delta})]$$
$$\Uparrow$$
$$-\frac{g^2}{4} f^{abe}f^{cde} A^\mu_a A^\nu_b A^c_\mu A^d_\nu$$



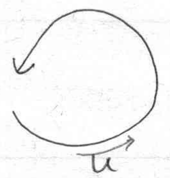
$$-ig\, q_\mu(-if_{abc}) \Leftarrow -g f_{abc}\, \partial^\mu (\chi^\dagger)^a A^b_\mu \chi^c$$

Note: energy-momentum conservation has to be imposed at each vertex.

Mnemonic rule for gluon vertices α ——— β $= g_{\alpha\beta}$



• Loops and problems

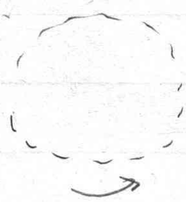


k

$$-\delta_{ii}\int\frac{d^4k}{(2\pi)^4}$$

k

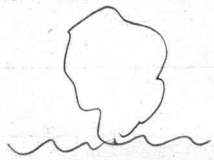
$$\delta_{aa}\int\frac{d^4k}{(2\pi)^4}$$



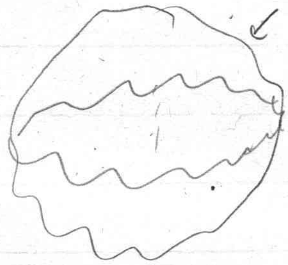
k

$$-\delta_{aa}\int\frac{d^4k}{2\pi^4}$$

Symmetry factors



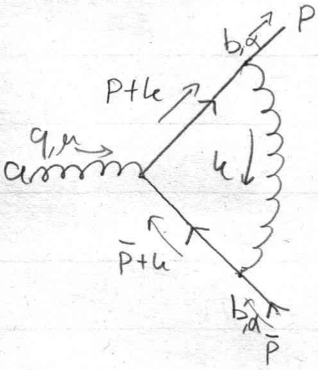
$\frac{1}{2!}$



m gluon propagators

$$=\frac{1}{m!}$$

Exercise: one-loop vertex in "Feynman gauge" $\lambda=1$



$$-\frac{1}{2N_c}t^a$$
$$\parallel$$

$$=(-ig)^3\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i\varepsilon}\gamma^\alpha\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2+i\varepsilon}\gamma_\mu\frac{i(\bar{\not{p}}+\not{k}+m)}{(\bar{p}+k)^2-m^2+i\varepsilon}\gamma_\alpha\, t^b t^a t^b =$$

$$=(-ig)\,i^3\left(-\frac{1}{2N_c}\right)t^a\int\frac{d^4k}{(2\pi)^4}\frac{\gamma^\alpha(\not{p}+\not{k}+m)\gamma_\mu(\bar{\not{p}}+\not{k}+m)\gamma_\alpha}{(k^2+i\varepsilon)[(p+k)^2-m^2+i\varepsilon][(\bar{p}+k)^2-m^2+i\varepsilon]}$$

Two problematic momentum regions

1) Large k $\Rightarrow$ $\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^4}$ logarithmic <u>ultra-violet divergence</u>

2) Small k, external quarks on-shell $p^2=\bar{p}^2=m^2$

$$\int\frac{d^4k}{(2\pi)^4}\frac{\gamma^\alpha(\not{p}+\not{k}+m)\gamma_\mu(\bar{\not{p}}+\not{k}+m)\gamma_\alpha}{(k^2+i\varepsilon)[(p+k)^2-m^2+i\varepsilon][(\bar{p}+k)^2-m^2+i\varepsilon]}\simeq\int\frac{d^4k}{(2\pi)^4}\frac{\gamma^\alpha(\not{p}+m)\gamma_\mu(\bar{\not{p}}+m)\gamma_\alpha}{(k^2+i\varepsilon)(2pk+i\varepsilon)(2k\bar{p}+i\varepsilon)}$$

Logarithmic <u>infrared divergence</u>