The QCD Lagrangian

We want to construct a quantum field theory for quarks starting from the information extracted from experiments

1) Quarks are fermions and exist in six "flavours" $f = u, d, c, s, t, b$

A good starting point is the Dirac free Lagrangian density

$$\mathcal{L}(x) = \sum_f \bar{\psi}_f(x)\,(i\partial\!\!\!/ - m_f)\,\psi_f(x) \qquad \partial\!\!\!/ \equiv \gamma^\mu \partial_\mu$$

2) A quark of each flavour has an additional quantum number, the "colour" $i = r, g, b$

$$\mathcal{L} = \sum_f \bar{\psi}_f^{\,i}\,(i\partial\!\!\!/ - m_f)\,\delta_{ij}\,\psi_f^{\,j}$$

this Lagrangian has a "global" $SU(N_c)$ symmetry $N_c = 3$

$$\psi_i(x) \to U_{ij}\,\psi_j(x) \quad \text{where} \quad SU(N_c) \ni U_{ij} = \left[e^{-i\theta^a t^a}\right]_{ij}$$

The group $SU(N_c)$ depends on $N_c^2 - 1$ <u>real</u> parameters. The matrices t^a_{ij} $a = 1, \dots, N_c^2 - 1$ are the <u>generators</u> of the "fundamental" representation of $SU(N_c)$, to which the quark field ψ_i belongs.

The properties of the generators t^a follow from the properties of the matrices $U \in SU(N_c)$

- U is unitary $\Rightarrow U^\dagger U = U U^\dagger = \mathbb{1} \Rightarrow (t^a)^\dagger = t^a$ <u>hermitian</u>
- $\det U = 1 \Rightarrow \det U = e^{i\theta^a \mathrm{Tr}\, t^a} = 1 \Rightarrow \mathrm{Tr}(t^a) = 0$ traceless

3) Since $SU(N_c)$ is an exact symmetry we can "gauge" it by requiring invariance under a "local" $SU(N_c)$, a.k.a. "gauge" transformation

$$\psi(x) \to U(x)\,\psi(x) \quad \text{where} \quad U(x) = e^{-i\theta^a(x)\,t^a}$$

The Lagrangian gets modified as follows

$$\mathcal{L} \to \mathcal{L} + i\,\bar{\psi}(x)\,U^\dagger(x)\,(\not{\partial}\,U(x))\,\psi(x)$$

"Gauge" invariance of $\mathcal{L}$ is restored after promoting the derivative ∂_μ to a "covariant" derivative D_μ which, as an operator, has the property

$$D_\mu\,U(x) = U(x)\,D_\mu$$

This is achieved by introducing a vector field $A_\mu(x)$ and defining

$$D_\mu \equiv \partial_\mu + ig\,A_\mu(x) \quad \text{where} \quad A_\mu(x) \text{ is a } N_c \times N_c \text{ matrix}$$

The desired tranformation properties of D_μ are obtained by requiring that A_μ transforms as follows

$$A_\mu(x) \to U(x)\,A_\mu(x)\,U^\dagger(x) + \frac{i}{g}(\partial_\mu U(x))\,U^\dagger(x)$$

Since $i(\partial_\mu U(x))\,U^\dagger(x) = \partial_\mu\theta^a(x)\,t^a$, $A_\mu(x)$ has to be a linear combination of the generators t^a

$$A_\mu(x) = A_\mu^a(x)\,t^a \Rightarrow N_c^2 - 1 \text{ "gluons" } A_\mu^a(x)$$

Dynamics of the gluon fields

The generators t^a are closed under commutation. They form an algebra named "the Lie algebra" of the group $SU(N_c)$

$$[t^a, t^b] = i f^{abc} t^c$$

The real numbers f^{abc} are the "structure constants" of $SU(N_c)$. As a tensor f^{abc} is totally antisymmetric and satisfies the Jacobi identity

$$f^{abc} f^{cde} + f^{aec} f^{cbd} + f^{adc} f^{ceb} = 0$$

The matrices t^a are normalised as follows

$$\mathrm{Tr}\,[t^a t^b] = T_F\, \delta^{ab} \qquad \text{where } T_F = 1/2$$

(Note: T_F is referred usually as T_R, here the label F refers to the "fundamental" representation to which t^a belong)

We can then work out how A^a_μ transform under an infinitesimal gauge transformation

$$U(x) = e^{-i\theta^a(x) t^a} \simeq \mathbb{1} - i\,\theta^a(x)\, t^a$$

$$A^a_\mu(x)\, t^a \to \overbrace{(\mathbb{1} - i\theta^c t^c)}^{U}(A^b_\mu t^b)\overbrace{(\mathbb{1} + i\theta^c t^c)}^{U^\dagger} + \frac{1}{g}\partial_\mu \theta^a t^a =$$

$$= A^a_\mu t^a + \frac{1}{g}\left[\partial_\mu \theta^a - g f^{abc} A^b_\mu \theta^c\right] t^a$$

We can now give a dynamics to A_μ^a by constructing the commutator of two covariant derivatives, the "field strength" $F_{\mu\nu}$

$$F_{\mu\nu} \equiv -\frac{i}{g}\left[D_\mu, D_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\left[A_\mu, A_\nu\right]$$

The tensor $F_{\mu\nu}$ is not gauge invariant, but rather transforms as

$$F_{\mu\nu}(x) \to U(x)\, F_{\mu\nu}(x)\, U^\dagger(x)$$

This is in contrast with the QED case, where $F_{\mu\nu}$ was gauge invariant. It's meaning is that gluons themselves interact via the "colour force".

The transformation rule of $F_{\mu\nu}$ is that of a vector belonging to the "adjoint" representation of $SU(N_c)$.
$F_{\mu\nu}(x)$ belongs to the Lie algebra of $SU(N_c)$, $F_{\mu\nu} = F_{\mu\nu}^a t^a$, where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c$$

After an infinitesimal gauge transformation

$$F_{\mu\nu}^a t^a \to (\mathbb{1} - i\theta^c t^c)\, F_{\mu\nu}^b t^b (\mathbb{1} + i\theta^c t^c) =$$

$$= F_{\mu\nu}^a t^a + f^{cba} t^a F_{\mu\nu}^b \theta^c = t^a\left(\delta^{ab} - i(-if^{cab})\,\theta^c\right) F_{\mu\nu}^b$$

Define now $(T^a)_{bc} = -if^{abc} = if^{bac}$ and obtain

$$F_{\mu\nu}^a \to \left(\delta^{ab} - i\,\theta^c (T^c)^{ab}\right) F_{\mu\nu}^b$$

This implies that the matrices $(T^a)_{bc}$ are the generators of the adjoint representation.

The Jacobi identity is just the commutation rule for the generators T^a

$$[T^a, T^b] = i f^{abc} T^c$$

The transformation rule for the gluon fields A_μ^a can now be written in terms of the generators T^a as follows:

$$(A_\mu^\theta)^a = A_\mu^a + \frac{1}{g} D_\mu^{ab} \theta^b$$

where D_μ^{ab} is the covariant derivative in the adjoint representation

$$D_\mu^{ab} = \partial_\mu \delta^{ab} + ig (T^c)^{ab} A_\mu^c$$

Note that the transformation rule for A_μ^a does not depend on the representation of the quark field ψ

We are finally able to construct a gauge invariant Lagrangiant for the gluon fields

$$\mathcal{L} = -\frac{1}{2T_F} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{4} (F_{\mu\nu}^a F^{\mu\nu a}) =$$

$$= -\frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) +$$

$$+ \frac{g}{2} f^{abc} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) A^{\mu b} A^{\nu c} +$$

$$- \frac{g^2}{4} f^{abc} f^{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e}$$

Note: gluons interact with each other!

Representations of $SU(N_c)$

- Fundamental representation 3 (quarks)

$$\psi'_i = U_{ij}\psi_j = \left(e^{-i\theta^a t^a}\right)_{ij}\psi_j \simeq \left(\delta_{ij} - i\theta^a t^a_{ij}\right)\psi_j$$

- Conjugate representation $\bar{3}$ (antiquarks)

$$\bar{\psi}'_i = \bar{\psi}_j U^\dagger_{ji} = U^*_{ij}\bar{\psi}_j = \left(e^{+i\theta^a (t^a)^T}\right)_{ij}\bar{\psi}_j \simeq \left(\delta_{ij} - i\theta^a(-t^T_a)_{ij}\right)\bar{\psi}_j$$

The generators of the conjugate representation are $\bar{t}_a = -t^T_a$

- Adjoint representation 8 (gluons)

$$A'_\mu = U A_\mu U^\dagger = e^{-i\theta^c t^c} A^b_\mu t^b e^{i\theta^c t^c} \simeq t^a\left(\delta^{ab} - i\theta^c(-if^{cab})\right)A^b_\mu$$

so that the generators of the adjoint representation are $(T^a)_{bc} = -if^{abc}$

- Higher representations: they can be constructed by taking tensor products of the fundamental and conjugate representation

$$T^{i_1\ldots i_n}_{j_1\ldots j_m} = \psi^{i_1}\ldots\psi^{i_n}\bar{\psi}_{j_1}\ldots\bar{\psi}_{j_m} \equiv \underbrace{3\otimes 3\otimes\ldots\otimes 3}_{n\ \text{times}}\otimes\underbrace{\bar{3}\otimes\bar{3}\otimes\ldots\otimes\bar{3}}_{m\ \text{times}}$$

The vector space spanned by these tensors can be decomposed into the direct sum of irreducible representations.

A representation of $SU(3)$ is irreducible if

$$\delta^j_i\, T^{i_1\ldots i\ldots i_n}_{j_1\ldots j\ldots j_n} = \epsilon^{ijk}\, T^{i_1\ldots i_n}_{j_1\ldots i\, j\, k\ldots j_n} = \epsilon_{ijk}\, T^{i_1\ldots i\, j\, k\ldots i_n}_{j_1\ldots j_n} = 0$$

Example 1: $3 \otimes \bar{3} = 1 \oplus 8$

$$\psi^i \bar{\psi}_j = \frac{1}{3}\delta^i_j (\psi^k \bar{\psi}_k) + \left[\psi^i \bar{\psi}_j - \frac{1}{3}\delta^i_j (\psi^k \bar{\psi}_k)\right] = A^i_j + B^i_j$$

It is straightforward to verify that A^i_j has only one independent component, an transforms trivially under $SU(N_c)$, while B^i_j has $N_c^2 - 1$ component and transforms according to the adjoint representation

Example 2: $3 \otimes 3 = 6 \oplus \bar{3}$

$$\psi^i \psi^j = \frac{1}{2}(\psi^i \psi^j - \psi^j \psi^i) + \frac{1}{2}(\psi^i \psi^j + \psi^j \psi^i) = A^{ij} + B^{ij}$$

For $N_c = 3$ A^{ij} has 3 independent component and transforms according to the conjugate representation, while B^{ij} has 6 independent components and spans the vector space of the new representation 6.

- Commutation rules, quadratic Casimir, etc.

The generators of each representation $T^a(R)$ satisfy the $SU(N_c)$ commutation rules

$$[T^a(R), T^b(R)] = i f^{abc} T^c(R)$$

This implies that the quadratic Casimir operator $C_2(R) = \sum_a T^a(R) T^b(R)$ commutes with all the generators, and therefore $C_2(R) = C_R \mathbb{1}_R$

The matrices $T^a(R)$ are normalised as $Tr[T^a(R) T^b(R)] = T_R \delta^{ab}$
Since higher representations are constructed from the fundamental representation, once T_F is specified, all other T_R can be computed

Graphical representation of SU(N_c) algebra

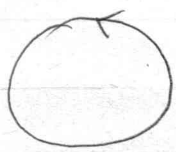
It is useful to have a graphical representation of colour matrices, because these always appear in the perturbative calculation of QCD amplitudes. Our representation will attribute to a graph a colour factor that is the same that will appear in Feynman diagrams

- Identity matrices $\Rightarrow$ propagators

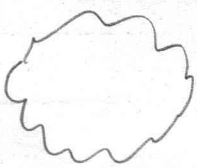
δ_{ij}

δ_{ab}

- Traces $\Rightarrow$ loops

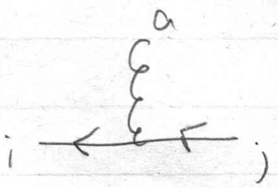


$\mathrm{Tr}(\mathbb{1}_F) = \delta_{ii} = N_c$

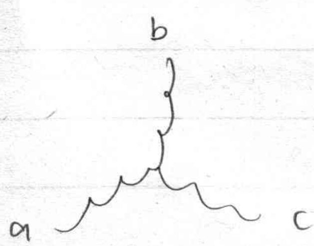


$\mathrm{Tr}(\mathbb{1}_A) = \delta_{aa} = N_c^2 - 1$

- Generators $\Rightarrow$ interaction vertices

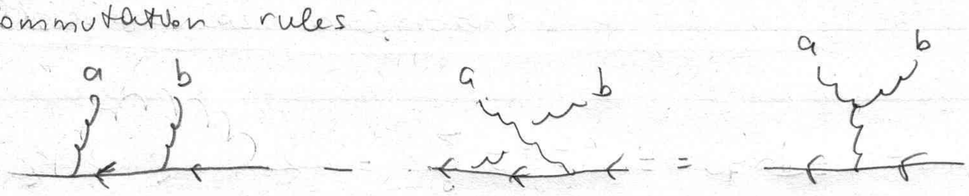


$(t^a)_{ij}$

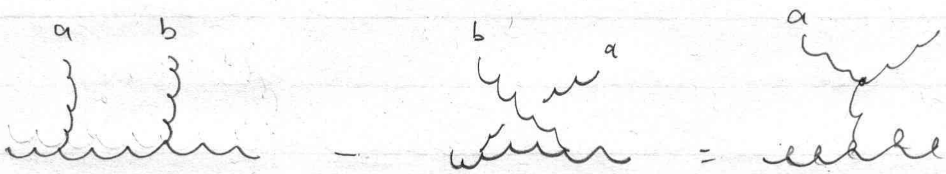


$i f_{abc} = (T^b)_{ac}$

- Commutation rules

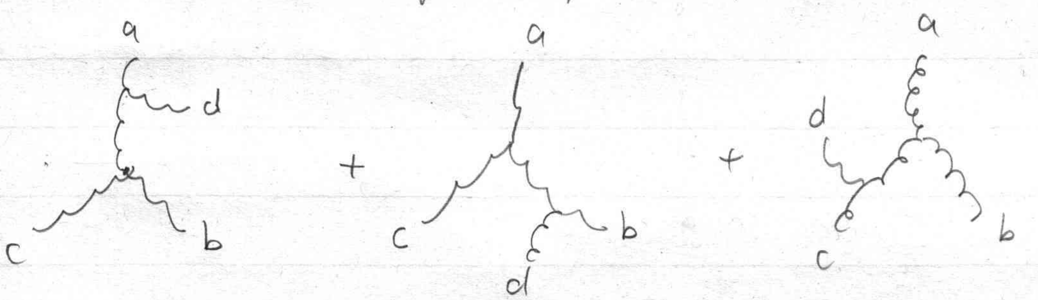


$[t^a,t^b]=if^{abc}t^c$



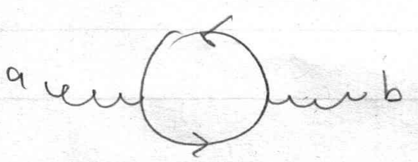
$[T^a,T^b]=if^{abc}T^c$

Note: this last equality corresponds to the Jacobi identity



$= 0$

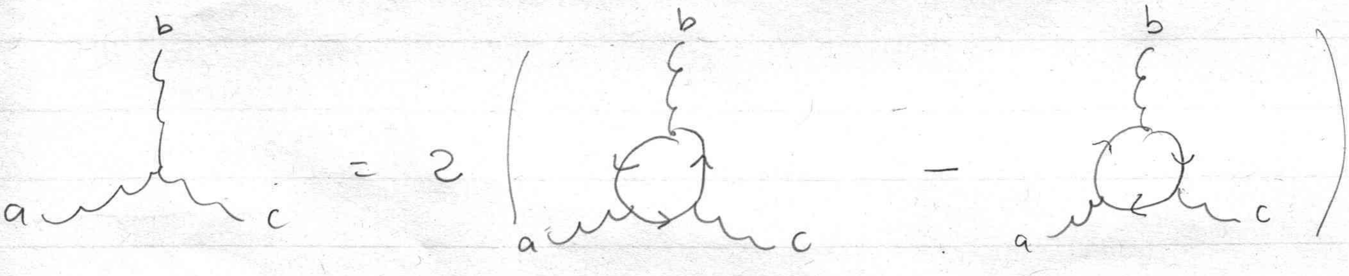
- Normalisation of generators



$= \mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab} = \frac{1}{2}$

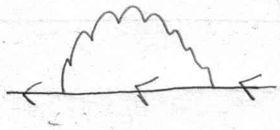
This gives a useful representation of if_{abc}. From the definition

$$[t^a,t^b]=if^{abd}t^d \Rightarrow if^{abc} = 2\,\mathrm{Tr}([t^a,t^b]\,t^c)$$

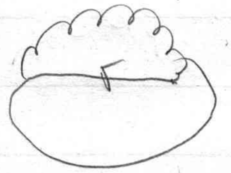


$= \mathrm{Tr}(t^a) = 0$

• Quadratic Casimir operators

 $= \sum_a t^a t^a = C_F \mathbb{1}_F = C_F$ 

To compute C_F we trace this expression

 $= C_F N_c = \frac{1}{2}$  $= \frac{N_c^2-1}{2} \Rightarrow C_F = \frac{N_c^2-1}{2N_c}$

 $= \sum_c (T^c)_{ae} (T^c)_{eb} = C_A (\mathbb{1}_A)_{ab} = C_A$ a ~~~~ b $= = \mathrm{Tr}(T^a T^b)$

To prove that $C_A = N_c$ we have to go through the Fierz identity

• Fierz identity

It is a completeness relation stemming from the fact that any $N_c \times N_c$ matrix can be written as a combination of the identity and of the generators of the fundamental representation t^a

$$A_{ij} = A_0 \delta_{ij} + A_a t^a = \frac{\mathrm{Tr} A}{N_c} \delta_{ij} + 2 \mathrm{Tr}(A \cdot t^a) t^a_{ij} =$$

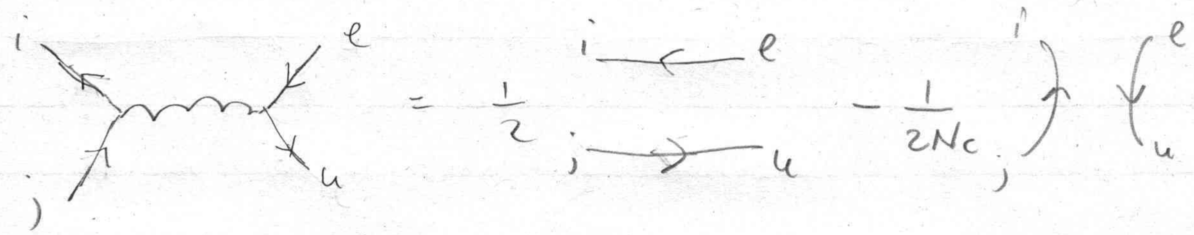
Written with full indices displayed

$$A_{ke} \delta_{ik} \delta_{je} = A_{ke} \left(\frac{1}{N_c} \delta_{ke} \delta_{ij} + 2 t^a_{ek} t^a_{ij} \right)$$

This gives

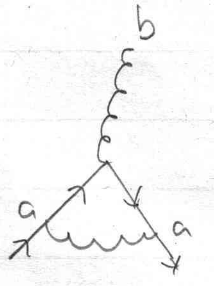
$$t^a_{ij} t^a_{ke} = \frac{1}{2} \delta_{ie} \delta_{jk} - \frac{1}{2N_c} \delta_{ij} \delta_{ke}$$

Pictorial representation of the Fierz identity



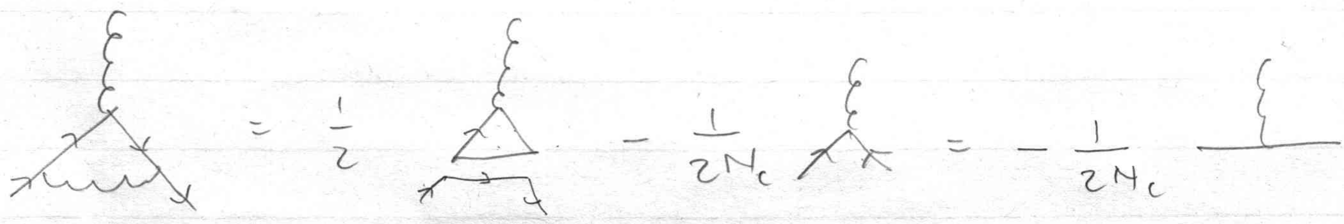
In the limit of large N_c, the gluon can be represented as a quark-antiquark line

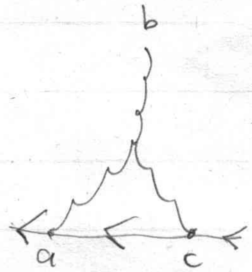
- Quark gluon vertex corrections



$= t^a t^b t^a = -\frac{1}{2N_c} t^b$

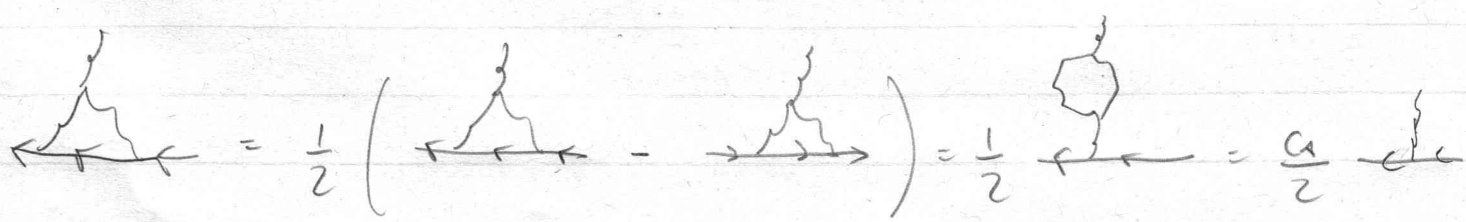
Using Fierz identity



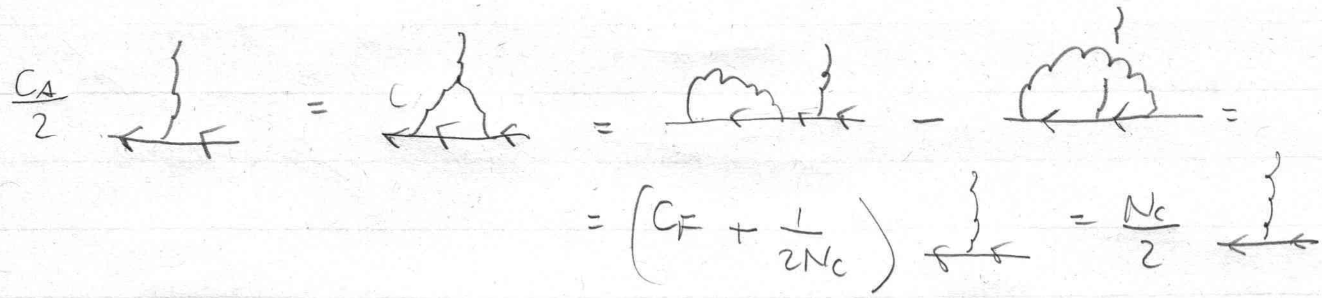


$= i f_{abc} t^a t^c = \frac{C_A}{2} t^b$

Using commutation rules



Exercise: show that $C_A = N_c$



$$\frac{C_A}{2} \; = \; = \; - \; = \; = \left(C_F + \frac{1}{2N_c}\right) \; = \frac{N_c}{2}$$

Comparing these two expressions we obtain $C_A = N_c$

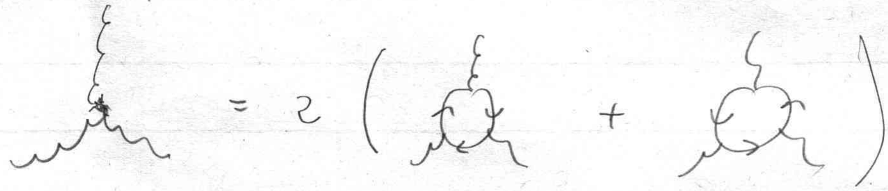
- Anticommutator between t^a

$$\{t_a, t_b\} = \frac{1}{N_c}\delta_{ab} + d_{abc}\, t^c$$

d_{abc} is a tensor that is symmetric in all its indices. By performing suitable traces one finds

$$d_{abc} = 2\,\mathrm{Tr}\left(t^a t^b t^c + t^b t^a t^c\right)$$

Pictorially



$$= 2\left(\; + \; \right)$$

Exercise: show the two main properties of d_{abc}

1) $d_{aac} = 0$

2) $d_{acd}\, d_{bcd} = \dfrac{N_c^2 - 4}{N_c}\,\delta_{ab}$