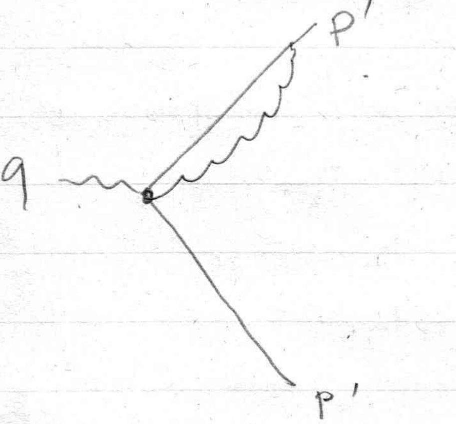
Coleman - Norton picture of pinch surfaces

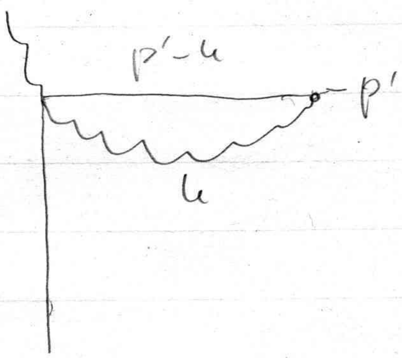
Suppose that a vertex creates particles at a position x^μ with momenta l_i^μ. If we interpret α_i as a time, in the Landau equation

$$\sum_j \eta_{ij} (\alpha_j l_j^\mu) = 0$$

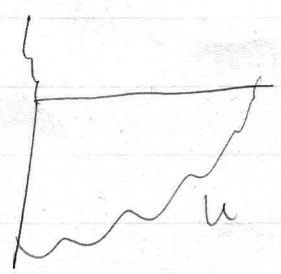
the quantity $\eta_{ij}\alpha_j l_j^\mu = \Delta\tau \cdot v_j^\mu = \Delta x_j^\mu$ represents the displacements of particle j from the position x^μ after a time $\Delta\tau$.
So the second Landau equation simply means $\sum_j \Delta x_j = 0$.
This means that after a time $\Delta\tau$ the particles have met again in a new vertex.

To any of the solutions of the Landau equations there corresponds a reduced diagram which admits a physical interpretation.
For instance for the vertex the collinear solutions correspond to the following reduced diagrams





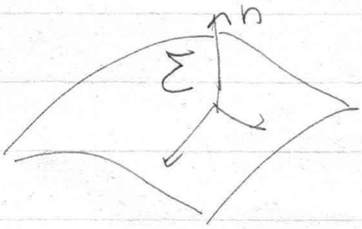
while soft diagrams are represented as lines joining the two hard partons



We can also have hard diagrams where all lines are off shell

Pinch surfaces and infrared/collinear singularities

Landau equations are necessary conditions for divergence of Feyman integrals. To see whether divergences occur one needs to perform the integral in the neighbourhood of a pinch singularity.



For each pinch surface Σ, one introduces internal coordinates in the surface and normal coordinates.

The divergence is evaluated by integrating over normal coordinates

Example 1: soft singularity

Pinch surface: $k^\mu = 0$ Normal coordinates: k^μ

Example 2: collinear singularity

Pinch surface: $k^2 = (p-k)^2 = 0$ $\quad k^\mu = \xi p^\mu$ $\quad p^\mu = \delta^\mu_+$

Internal coordinates: k^+, φ → azimuthal angle around direction of p_z

Normal coordinates: k^-, $k_\perp$

For each neighbourhood of a pinch surface one has then a contribution $4S + 2C$ from ~~soft~~ and collinear loops.

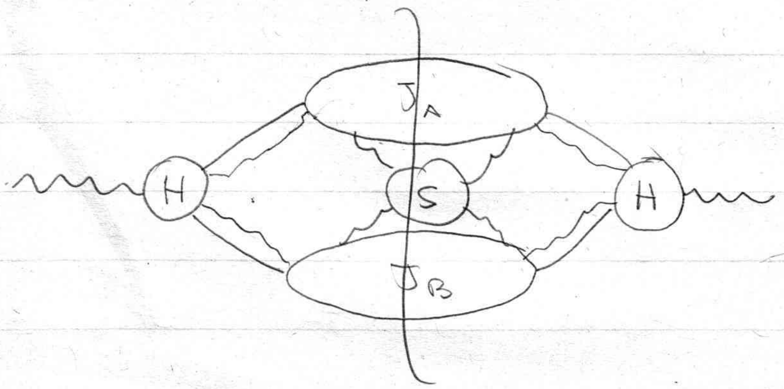
One then has a procedure to evaluate what is the contribution to the numerator and the denominator to the integrand. This power counting is based on the soft approximation and, graph by graph, depends on the gauge. Roughly, for a reduced graph G

$$\lambda(G) = [4S - 2\underset{\text{soft gluons}}{N_G} - \underset{\text{hard fermions}}{N_F}] + 2C - \underset{\text{collinear lines}}{I_C} + \text{Numerator and vertex contributions}$$

If $\lambda(G) \leq 0$ the graph is infrared divergent

Leading regions in Feynman gauge

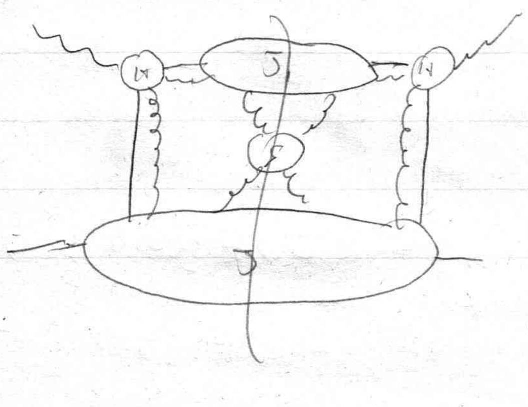
The reduced diagrams that gives to infrared and/or collinear singularities correspond to the so-called "leading regions" of integration.
Performing the Coleman-Norton analysis together with power counting, one finds that in Feynman gauge, the leading region of a generic cut graph in e^+e^- annihilation are as follows

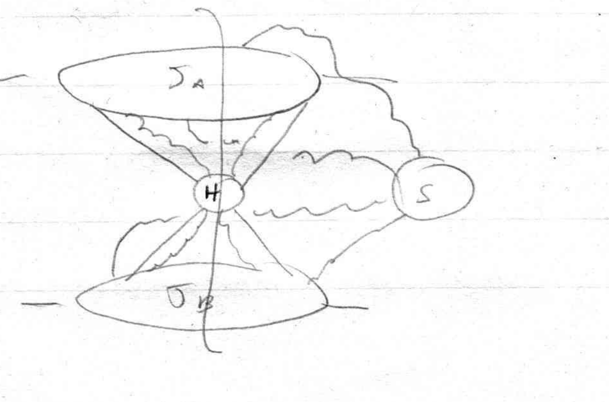


We recognise three sets of contributions
1) Hard: all lines are off shell
2) Jet: parallel lightlike lines all pointing in the same direction
3) Soft: zero-momentum lines

with the following properties, at the pinch surface:
1) Soft lines attach only to the jets and not to the hard subdiagrams
2) The jets attach to the hard subdiagram via a single physically polarised line. An arbitrary number of collinear gluons can attach to the hard subdiagram, but only with longitudinal polarisations.

The same kind of power counting holds for analogous contributions to DIS or DY at a pinch surface

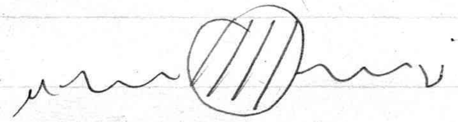




Exercise on power counting: cancellation of soft

The total cross section ~~in a~~ $e^+e^- \to$ hadrons, is related via unitarity to the imaginary part of the photon propagator

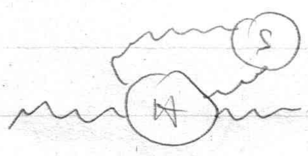
$$\sigma(e^+e^- \to \text{hadrons}) \propto L^{\mu\nu} W_{\mu\nu} \qquad W_{\mu\nu} \propto \operatorname{Im} \Pi_{\mu\nu}(q^2)$$

$\Pi_{\mu\nu}(q^2) =$  $\qquad q^2 > 0$

Π

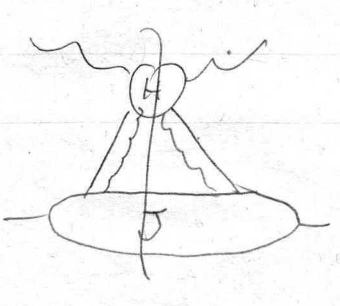
We go to a reference frame in which $q = (q_0, \vec{0})$ and look for pinch surfaces. Collinear surfaces would require two lines emerging from the vacuum ~~and~~ at a given time, travelling at the speed of light and then reuniting at a different time. But this is not possible because two particles created from the vacuum will have opposite three-momenta from momentum conservation. They will then travel in opposite directions and will never meet each other again the future.

The only pinch surfaces are soft lines attaching to the hard vertex

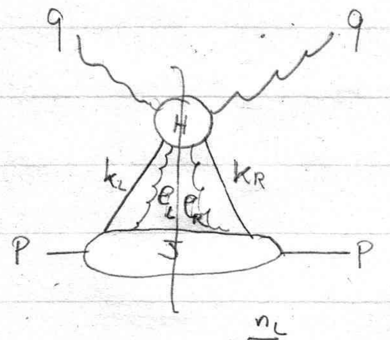


However, since we have no fermion lines attached to the hard diagram this contribution is suppressed by power counting. We have then the IR and collinear safety of the e^+e^- total cross section.

The same argument applies to DIS cross section, related to the imaginary part of a forward $\gamma^* q$ amplitude



Factorisation for DIS



$e_L = \{e_i\}_{i=1\dots n_L}$ at the left of the cut

$e_R = \{e'_j\}_{j=1\dots n_R}$ at the right of the cut

$$k_L = k - \sum_{i=1}^{n_L} e_i \qquad k_R = k - \sum_{j=1}^{n_R} e'_j .$$

k is an internal loop momentum and we have to integrate over it.
A generic graph in the neighbourhood of a pinch surface has the form

$$G^{\ell} = \sum_{\text{cuts}} G^{(\text{cut})} = \int \frac{d^4k}{(2\pi)^4} \int \prod_i \frac{d^4 e_i}{(2\pi)^4} \int \prod_j \frac{d^4 e'_j}{(2\pi)^4} \times$$

polarisations of $\{e_i\}, \{e'_j\}$

$$\times \sum_{C_H} H^{(C_H)}\left(q^\nu, k^\mu - \sum_i e_i^{\mu}, k^\mu - \sum_j e'^{\mu}_j, \{e_i^{\alpha_i}\}, \{e'^{\beta_j}_j\}\right)^{\{\mu_i,\nu_j\}}_{\eta,\eta'}$$

$\leftarrow$ polarisation of k_L, k_R

$$\times \sum_{C_J} J^{(C_J)}\left(p^\nu; k^\mu - \sum_i e_i^{\mu}, k^\mu - \sum_j e'^{\mu}_j, \{e_i^{\alpha_i}\}, \{e'^{\beta_j}_j\}\right)^{\eta,\eta'}_{\{\mu_i,\nu_j\}}$$

We now perform various approximations, corresponding to those that give a leading power behaviour for the above contributions

1) The hard diagrams depend only on the large "+" component of the momenta entering in it

$$p^\mu = p^+ v^\mu \qquad k^\mu = k^+ v^\mu \qquad e_i^\mu = e_i^+ v^\mu$$

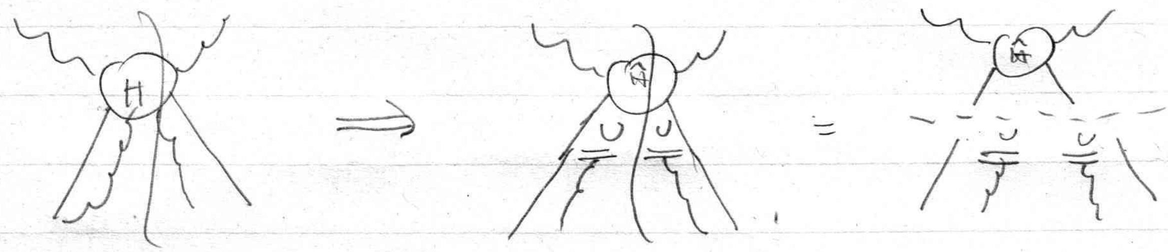
$$H\left(q^\nu, k^\mu - \sum_i e_i^\mu, k^\mu - \sum_j e'^\mu_j, \{e_i^{\alpha_i}\}, \{e'^{\beta_j}_j\}\right) \simeq$$

$$\simeq \bar{H}\left(q^\nu, (k^+ - \sum_i e_i^+) v^\mu, (k^+ - \sum_j e'^+_j) v^\mu, \{e_i^+ v^{\alpha_i}\}, \{e'^+_j v^{\beta_j}\}\right)_{\eta,\eta'} \prod_i u^{\mu_i} \prod_j u^{\nu_j}$$

where u^μ is such that $u \cdot v = 1$ and

$$\bar{H}(\;)_{\eta,\eta'} = H(\;)^{\{\gamma_i,\delta_j\}}_{\eta,\eta'} \prod_i v_{\gamma_i} \prod_j v_{\delta_j}$$

2) Gluons enter in H only with polarisation along v^μ, which is effective longitudinal polarisation. Since they are also coupled only to the "$-$" component of all momenta in the hard diagram, they couple effectively to an eikonal line moving with four-velocity v^μ.



This is a consequence of gauge invariance and can be proven formally by repeated use of Ward identities.

$$H^-\left(q^\nu, (k^+ - \sum_i e_i^+)v^\mu, (k^+ - \sum_j e_j'^+)v^\nu, \{e_i^{\alpha_i}\}, \{e_j'^{\beta_j}\}\right)_{\eta,\eta'} \prod_i v^{\mu_i} \prod_j v^{\nu_j}$$

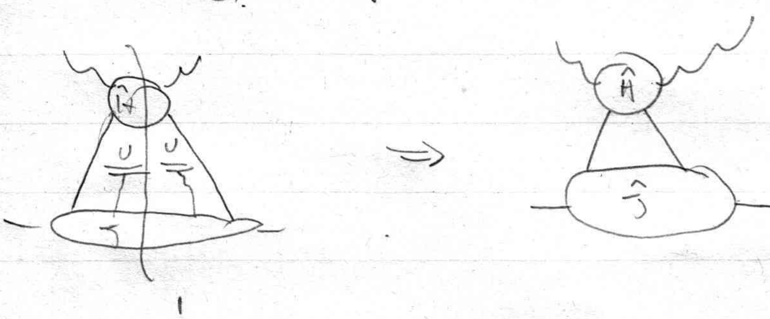
$$= \hat{H}(q^\nu, k^+ v^\mu)_{\eta,\eta'} \, \mathcal{E}(v, \{e_i^+\})^{\{\mu_i\}} \, \mathcal{E}^*(v, \{e_j'^+\})^{\{\nu_j\}}$$

3) Parton ~~model~~ interpretation. Introduce a new "jet function"

$$\hat{J}^{\eta,\eta'}(p^\nu, \xi) = \int \frac{d^4k}{(2\pi)^4} \, \delta\left(\xi - \frac{k\cdot v}{p\cdot v}\right) \cdot \int \prod_i \frac{d^4 e_i}{(2\pi)^4} \int \prod_j \frac{d^4 e_j'}{(2\pi)^4} \times$$

$$\times \, \mathcal{E}(v, \{e_i^+\})^{\{\mu_i\}} \, \mathcal{E}^*(v, \{e_j'^+\})^{\{\nu_j\}} \sum_{C_J} J^{(C_J)}\left(p^\nu, k^\mu - \sum_i e_i^\mu, k^\nu - \sum_j e_j'^\nu, \{e_i^{\alpha_i}\}, \{e_j'^{\beta_j}\}\right)^{\eta,\eta'}_{\{\mu_i,\nu_j\}}$$

we have, using the fact that a singular term is obtained only for $|\xi| \leq 1$



$$G = \int_{-1}^{1} d\xi \, \hat{H}(q^\nu, \xi p^\mu)_{\eta,\eta'} \, \hat{J}^{\eta,\eta'}(p^\nu, \xi)$$

4) Factorisation in spin space

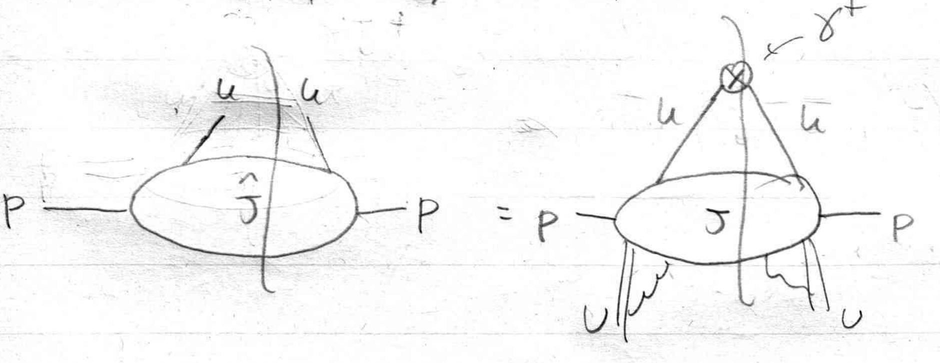
Since $\hat{J}^{\eta,\eta'}(p^\nu,\xi)$ depends only on the vector p^ν, its spin structure, in unpolarised scattering, can only be, for a leading region,

$$\hat{J}^{\eta,\eta'}(p^\nu,\xi) = (p^+\gamma^-)^{\eta,\eta'} g_{a/p}(\xi)$$

where $g_{a/p}(\xi) = \frac{1}{4p^+}\,\mathrm{Tr}\left(\gamma^+\hat{J}(p^\nu,\xi)\right)$ and hence

$$\int_{-1}^{1} d\xi\; \hat{H}(q,\xi p^+\cdot v)_{\eta,\eta'}\; \hat{J}^{\eta,\eta'}(p,\xi) = \int_{-1}^{1} d\xi\; H_a(q,\xi p^+ v,\mu)\, g_{a/p}(\xi,\mu)$$

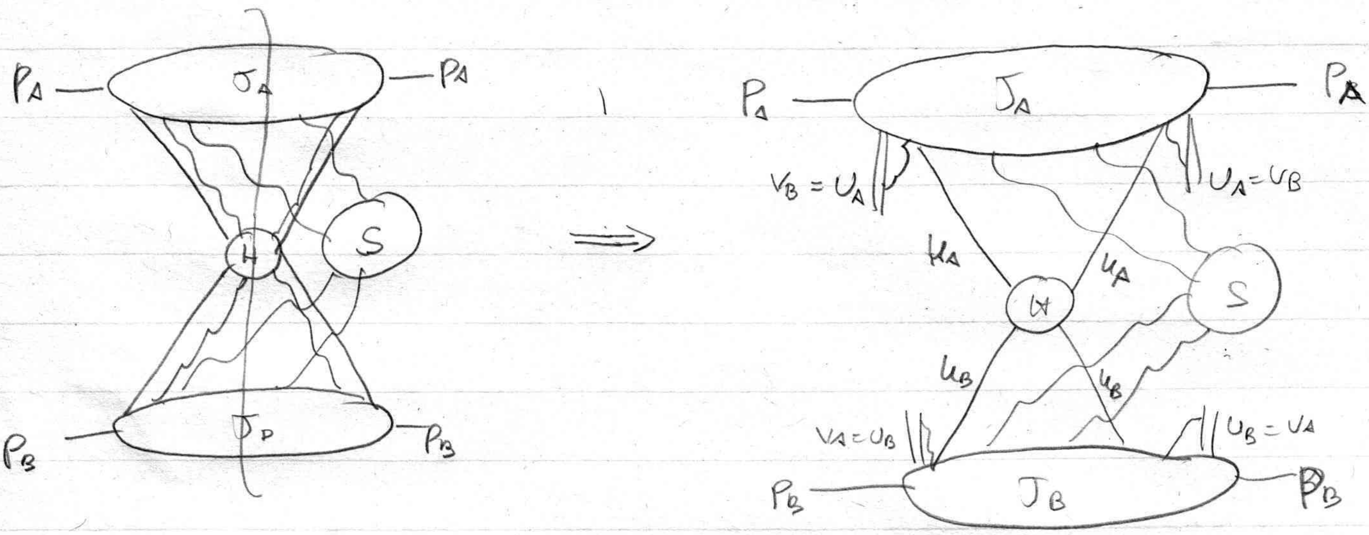
One can then refine the definition of $g_{a/p}$, but this is in all respect the density of parton u of type a in parton p. Notice that $\xi < 0$ corresponds to an antiquark. Neglecting numerical factors a parton distribution is obtained graphically as



$\propto \mathrm{Tr}(\gamma^+ J(p,\xi))$

Factorisation in Drell-Yan process (graphical considerations only)

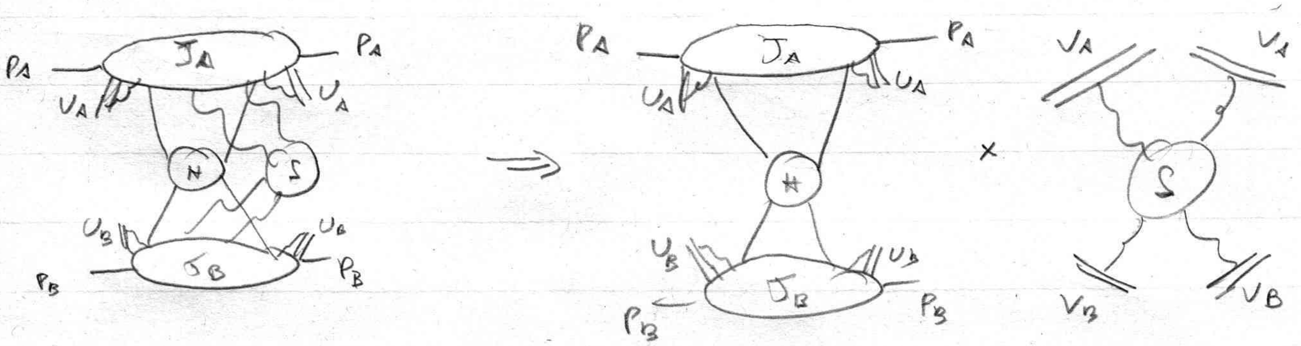
Leading regions for Drell-Yan process



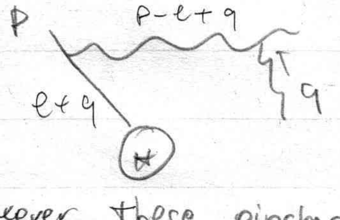
In order to have the same parton distributions as before we have to decouple the soft from the jets. The jet J_A is made up of lines ℓ_i quasi parallel to P_A, which couple to a soft gluon q with a denominator

$$\frac{1}{(\ell+q)^2+i\varepsilon} \simeq \frac{1}{\ell^2+2\ell^+q^- - 2\vec{\ell}_\perp\cdot\vec{q}_\perp - q_\perp^2+i\varepsilon} \simeq \frac{1}{2\ell^+q^-+i\varepsilon} \quad \text{for } |q^-| \gg \frac{|q_\perp^2+2\vec{\ell}_\perp\vec{q}_\perp-\ell^2|}{2\ell^+}$$

If this is valid the coupling of soft to a jet corresponds to an eikonal coupling to a line moving in the "+" direction, that of P_A.



This is possible as long as one can deform the q^- contour. This is spoiled by configurations like this, where the two poles in q^- plane are at



$$q^- = (p-\ell)^2 + 2(\vec{p}_\perp-\vec{\ell}_\perp)\cdot\vec{q}_\perp - q_\perp^2 + i\varepsilon$$

$$q^- = \frac{-\ell^2+2\vec{\ell}_\perp\cdot\vec{q}_\perp + q_\perp^2 - i\varepsilon}{2\ell^+}$$

However these pinched contributions cancel when one sums over all final state cuts.

KLN theorem and infrared and collinear safety

Cancellation of infrared and collinear singularities is ruled by the Kinoshita-Lee-Nauenberg theorem.

Consider a set of initial states $|a\rangle$ and final states $|b\rangle$.

The Hamiltonian $H = H_0 + H_I$ where H_0 is a free part and H_I an interaction part and $H_0|a\rangle = E_a|a\rangle$, $H_0|b\rangle = E_b|b\rangle$ with $E - \epsilon_0 < E_{a,b} < E + \epsilon_0$, i.e. $a, b \in D(E)$

The S matrix is defined in the interaction picture as

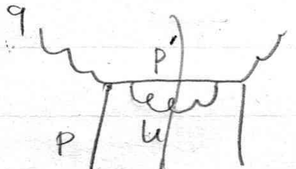
$$S = T e^{-i\int_{-\infty}^{+\infty} dt\, H_I(t)} = T e^{-i\int_{-\infty}^{+\infty} dt\, e^{iH_0 t} H_I e^{-iH_0 t}}$$

Then $\sum_{a\in D(E)}\sum_{b\in D(E)} |\langle b|S|a\rangle|^2$ is free from infrared and/or collinear divergences

In other words, cross sections are finite whenever one sums over degenerate states

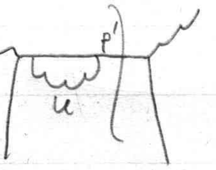
Example: DIS cross section (in physical gauge) $E^2 = (p+q)^2$

1) Final state interaction $k // p'$



$$\propto \int_0^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \int_0^1 dz\, P_{qq}(z)\, \delta((\underbrace{p+q}_{p'}-k)^2) \simeq$$

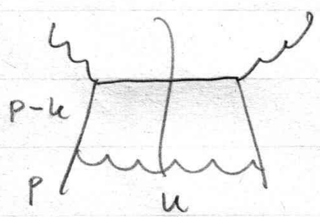
$$\simeq \frac{1}{2p\cdot q} \int_0^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \int_0^1 dz\, P_{qq}(z)\, \delta(-x + 1 \underbrace{-2(p+q)k}_{\ll 1,\ k \text{ collinear to } p'})$$



$$\propto \int_0^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \int_0^1 dz$$

The collinear singularity cancels, the final states $|p'\rangle$ and $|p'k\rangle$ with $k // p'$ are degenerate

2) Initial state interactions $k // p$ $\quad k \simeq (1-z)p$



$$\propto \int_0^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \int_0^1 dz\, P_{qq}(z)\, \delta(-x + 1 - \underbrace{\frac{2pk}{2q\cdot p}}_{\ll 1\ k//p} - \underbrace{\frac{2qk}{2q\cdot p}}_{\simeq (1-z)}) \simeq$$

$$\simeq \int_0^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \int_0^1 dz\, P_{qq}(z)\, \delta(x-z)$$

This state is not degenerate with the virtual which gives $\delta(x-1)$ contribution.
Lack of averaging over initial state leads to collinear singularities

Infrared and collinear safe observables

In the spirit of KLN theorem one invents observables where one can average over states, so that infrared and collinear singularities cancel. These are the so-called infrared and collinear safe observables. These observables:

1) Can be safely computed in perturbation theory
2) Corrections from parton to hadron level are suppressed by inverse powers of the hard scale

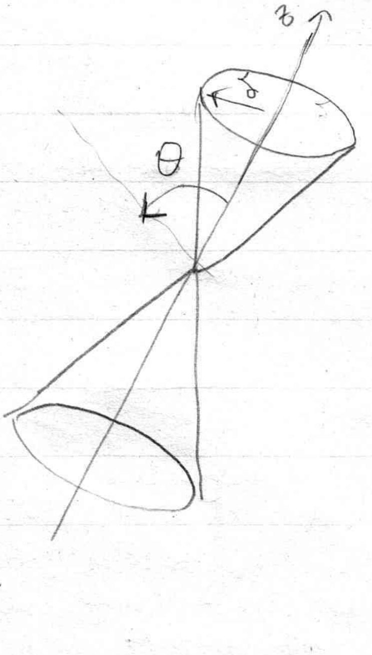
$$\sigma_{had}\left(\frac{Q}{\mu}, \alpha_s(\mu), v\right) = \bar{\sigma}(1, \alpha_s(Q), v) + O\left(\left(\frac{m_{had}}{Q}\right)^p\right)$$

The hadron mass $m_{had} = m_\pi, m_p, \ldots$ plays the role of the energy resolution Q_0 of the KLN theorem.

1) Total e^+e^- annihilation cross section: a state with no gluons give the same contribution as a state with an arbitrary number of gluons. All infrared and collinear singularities cancel completely between real and virtual.

2) Sterman-Weinberg jets: first example of an observable that is not fully inclusive.

In e^+e^- annihilation one defines a two-jet cross section by requiring that the total energy contained in two cones of opening angle δ is larger than $(1-\varepsilon)\cdot\sqrt{s}$



1) Collinear safety: after a collinear splitting the energy stays in the same jet ⇒ degenerate with state with no splittings

2) infrared safety: any soft emission does not change how the energy is distributed between the jets and the region outside the jets ⇒ degenerate with state with no emissions

Modern IRC safe observables: event shapes and jets

Roughly speaking, an IRC safe observable is insensitive to any number of soft emissions and collinear splitting. Degeneracy occurs between states with and without emissions/splittings.

Consider now a variable $V(q_1 \ldots q_n)$ where $\{q_i\}$ are final state momenta.

If $V(q_1 \ldots q_n)$ does not change by adding an arbitrary number of soft emissions and/or collinear splittings, the corresponding distribution

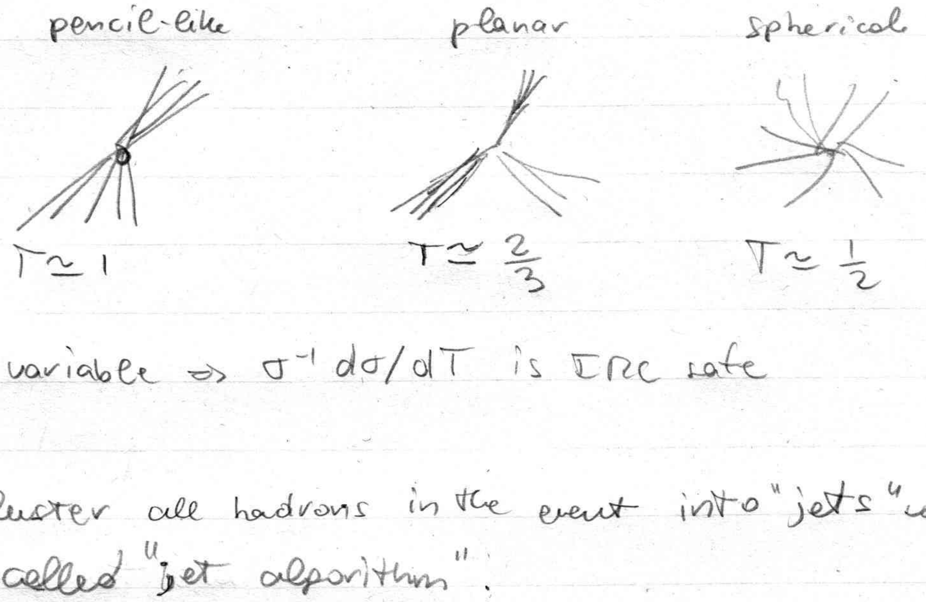
$$\frac{1}{\sigma}\frac{d\sigma}{dV} = \frac{1}{\sigma}\sum_{n=0}^{+\infty}\int \prod_i [dq_i] \frac{d\sigma}{[dq_1]\ldots[dq_n]}\, \delta(V - V(q_1 \ldots q_n))$$

is an IRC safe observable. Then $V(k_1 \ldots k_n)$ is referred to as an IRC safe variable.

1) Event shapes: $V(k_1 \ldots k_n)$ is a number that is related to the geometry of the event. Example, the thrust in e^+e^- annihilation

$$T = \max_{\vec{n}} \frac{\sum_i |\vec{q}_i \cdot \vec{n}|}{\sum_i |\vec{q}_i|}$$

$\frac{1}{2} < T < 1$



Thrust is an IRC safe variable $\Rightarrow$ $\sigma^{-1} d\sigma/dT$ is IRC safe

2) Jet cross sections: Cluster all hadrons in the event into "jets" with a definite procedure called "jet algorithm".

If the number of jets does not change after any number of soft emissions and/or collinear splittings, then jet cross sections, cross sections differential in momenta of jets, are infrared and collinear safe quantities.

Radiation suppression and Sudakov form factors

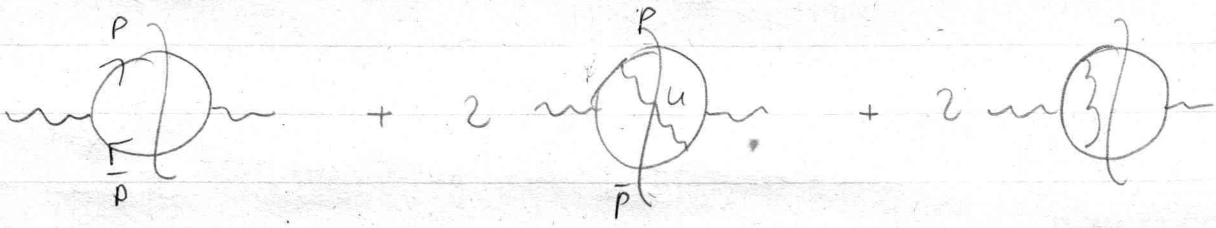
Consider the thrust distribution

$$\Sigma(\tau) = \int_{1-\tau}^{1} \frac{1}{\sigma}\frac{d\sigma}{dT}$$

For $\tau \ll 1$ this distribution gets contribution only from "pencil-like" events. In the region of small τ

$$1-T \simeq \sum_i \frac{\omega_i}{Q}\left(1-|\cos\theta_i|\right) \qquad Q = \sum_i \omega_i$$

We now compute $\Sigma(\tau)$ at one loop in the soft limit



$$\Sigma(\tau) = 1 + 2C_F g^2 \int [dk] \frac{p\bar{p}}{(pk)(k\bar{p})}\left[\Theta\left[\tau - \frac{\omega}{Q}\left(1-|\cos\theta|\right)\right] - 1\right] \simeq$$

$$\simeq 1 - C_F \frac{\alpha_s}{\pi} \ln^2\frac{1}{\tau}$$

1) Both real and virtual contributions are separately divergent but their total contribution is finite. Only virtual corrections survive the cancellation

2) For $\tau \to 0$ (perfect two-jet event) $\Sigma(\tau)$ diverges at one loop, however, the series can be resummed, thus giving

$$\Sigma(\tau) = e^{-C_F \frac{\alpha_s}{\pi} \ln^2\frac{1}{\tau}} \to 0 \qquad \tau \to 0$$

double logarithm, one for soft and the other for collinear singularity

Accelerated quarks are always accompanied by gluons. Probability of having a quark without accompanying radiation is zero

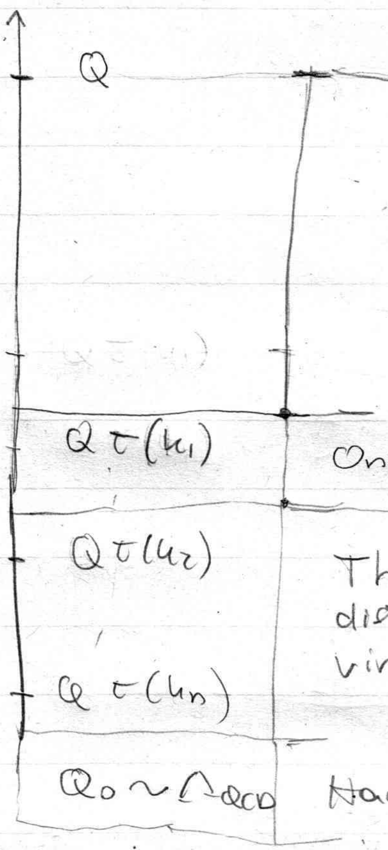
3) The thrust differential distribution is obtained by differentiating $\Sigma(\tau)$ with respect to τ

$$\frac{1}{\sigma}\frac{d\sigma}{d\tau} = \frac{1}{\tau}\left(2\alpha_s \frac{C_F}{\pi}\ln\frac{1}{\tau}\right) \times e^{-\alpha_s \frac{C_F}{\pi}\ln^2\frac{1}{\tau}} \times \left(1 + \mathcal{O}\left(\frac{Q_0}{\tau Q}\right)\right)$$

Emissions giving rise to this expression are soft and collinear emissions with $\tau(k_1) \gg \tau(k_2) \gg \ldots \gg \tau(k_n)$, where

$$\tau(k) := \frac{\omega}{Q}\left(1-|\cos\theta|\right) \qquad 1-T \simeq \sum_i \tau(k_i)$$

In this case $1-T \simeq \tau(k_1)$ and we have the following hierarchy of scales



Only virtual corrections, giving the probability of having no real emission with $\tau(k) > \tau$

$\Downarrow$

Sudakov form factor

$$e^{-\int [dk] M^2(k)\, \Theta(\tau(k)-\tau)} \simeq e^{-C_F\frac{\alpha_s}{\pi}\ln^2\frac{1}{\tau}}$$

One-gluon emission probability $\frac{1}{\tau}\left(2C_F\frac{\alpha_s}{\pi}\ln\frac{1}{\tau}\right)$

These emission do not contribute to the thrust distribution. Their contribution cancel completely with virtual corrections, giving $\left(1+\mathcal{O}\left(\frac{Q_0}{\tau Q}\right)\right)$

Hadronisation corrections