Infrared and collinear safety

Consider a generic partonic cross section $\sigma\left(\alpha_s(\mu), \frac{Q}{\mu}, \frac{\overline{m}}{\mu}\right)$. Let us assume that $\overline{m}$ is the running mass in some mass-independent renormalisation scheme (e.g. MS or $\overline{MS}$).

σ is infrared and collinear safe if and only if

$$\sigma\left(\alpha_s(\mu), \frac{Q}{\mu}, \frac{\overline{m}}{\mu}\right) = \overline{\sigma}\left(\alpha_s(\mu), \frac{Q}{\mu}\right) + O\left(\left(\frac{\overline{m}}{Q}\right)^p\right)$$

In this case we can safely take the $Q \to +\infty$ limit, since, in an asymptotically free theory $m \to 0$ for $Q \to +\infty$

If a quantity is not infrared and collinear safe, σ may have $\ln(\overline{m}^2/Q^2)$, so that the limit $Q \to +\infty$ cannot be taken. In principle one could consider taking the pole mass m, but even for heavy quarks with $m \gg \Lambda_{QCD}$, one can have large logarithms $\ln(m^2/Q^2)$ that need to be resummed at all orders

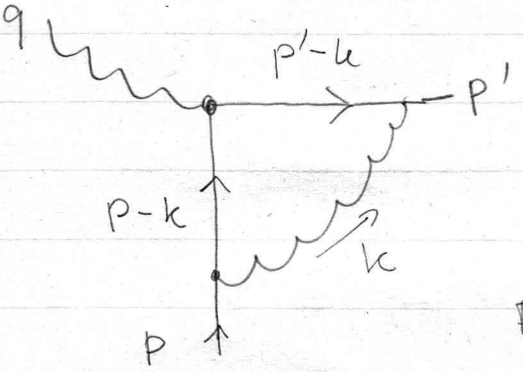
Factorisation is the statement that for infrared and collinear unsafe observables one can write the cross section as the convolution of a "partonic" cross section, finite in the limit $Q \to +\infty$, and a "process independent" parton distribution, containing all $\ln(m^2/Q^2)$

$$\sigma\left(\alpha_s(\mu), \frac{Q}{\mu}, \frac{\overline{m}}{\mu}\right) = \hat{\sigma}\left(\alpha_s(\mu), \frac{Q}{\mu}\right) \otimes f\left(\alpha_s(\mu), \frac{Q}{\mu}, \frac{\overline{m}}{\mu}\right) + O\left(\left(\frac{\overline{m}}{Q}\right)^p\right)$$

N.B.: these statements hold in perturbative QCD, independent of the fact that quarks are not observed as free states.

Infrared and collinear divergences

Consider as an example the basic diagram contributing to the DIS process, and the vertex correction to it



Feynman gauge Fermion propagators

$$(-ig)^2 C_F \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2+i\varepsilon} (-g_{\alpha\beta}) \, i^2 \, \frac{\bar{u}(p')\gamma^\alpha(\not p'-\not k+m)\gamma_\mu(\not p-\not k+m)\gamma_\beta u(p)}{[(p'-k)^2-m^2+i\varepsilon][(p-k)^2-m^2+i\varepsilon]} =$$

$$= -i C_F \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')\gamma^\alpha(\not p'-\not k+m)\gamma_\mu(\not p-\not k+m)\gamma_\alpha u(p)}{[k^2+i\varepsilon][2p'k-k^2-i\varepsilon][2pk-k^2-i\varepsilon]}$$

Let us consider a frame in which p and p' move in opposite directions

$$p = \frac{Q}{2}(1,0,0,1) \qquad p' = \frac{Q}{2}(1,0,0,-1) \Rightarrow q = (0,0,0,-Q)$$

It is useful to introduce light-cone coordinates. For each k, we fix two lightlike vectors v and u

$$v = \frac{1}{\sqrt{2}}(1,0,0,1) \qquad u = \frac{1}{\sqrt{2}}(1,0,0,-1) \qquad u \cdot v = 1$$

$$k^\mu = k^+ v^\mu + k^- u^\mu + k_\perp^\mu \qquad k^+ = k \cdot u \qquad k^- = k \cdot v \qquad k_\perp^\mu k_{\perp\mu} = -k_\perp^2 < 0$$

In our case $p^\mu = p^+ v^\mu$ and $(p')^\mu = (p')^- u^\mu$, and

$$k^2 = 2k^+k^- - k_\perp^2 \qquad 2pk = 2p^+k^- \qquad 2p'k = 2(p')^-k^+$$

$$d^4k = dk^+ \, dk^- \, d^2k_\perp$$

Consider now the integration region in which k is soft. Then

1) we can neglect all factors of $\not k$ in the numerator

2) we can approximate $2p^+k^- - k_\perp^2 - 2k^+k^- \simeq 2p^+k^-$

These are the configurations that give a divergent contribution to the integral. Indeed, using Dirac equation $(\not p - m)u(p) = \bar u(p')(\not p' - m) = 0$

$$-iC_F g^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar u(p')\gamma^\alpha(\not p'+m)\gamma_\mu(\not p+m)\gamma_\alpha u(p)}{[k^2+i\epsilon][2pk-i\epsilon][2p'k-i\epsilon]} =$$

$$= -iC_F g^2 \underbrace{\int \frac{d^4k}{(2\pi)^4}\frac{(p\cdot p')}{[k^2+i\epsilon][pk-i\epsilon][p'k-i\epsilon]}}_{\int \frac{d^4k}{k^4} \text{ logarithmically divergent}} \times \left[\bar u(p')\gamma_\mu u(p)\right]$$

Born vertex

In the region where k is soft (all its components vanish) the integral is logarithmically divergent.

If approximation 2) is valid, in order to have a divergence we need to have for each gluon line at least two fermion lines, naively the degree of divergence $\lambda(G)$, given S loop integrals, with N_G gluon lines and N_F fermion lines, in graph G, is

$$\lambda(G) = 4S - 2N_G - N_F$$ The graph diverges if $\lambda(G) \le 0$

If soft approximation is valid, soft singularities have the following properties

1) They factorise from the hard matrix element

2) Soft gluons are emitted with eikonal Feynman rules

u^μ  μ $\quad -ig\,\dfrac{u^\mu}{u\cdot k}$

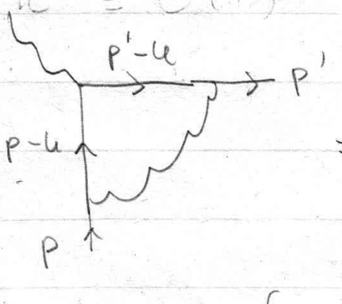
Notice that if $u^\mu = g^\mu_+$

$u\cdot k = u^+k^-$ $\quad u\cdot\varepsilon = u^+\varepsilon_-$

A gluon emitted with eikonal Feynman rules is longitudinally polarised in the direction opposite to its emitter

- Collinear divergences

We now consider the region in which k is hard but collinear to p, that is $k^+ = \xi p^+$ with $\xi = O(1)$. Neglecting the masses



$$= -i C_F g^2 \int \frac{dk^+ dk^- d^2k_\perp}{(2\pi)^4} \frac{\bar{u}(p')\gamma^\alpha(\not{p}'-\not{k})\gamma_\mu(\not{p}-\not{k})\gamma_\alpha u(p)}{[k^2+i\varepsilon][2pk-k^2-i\varepsilon][2p'k-k^2-i\varepsilon]} =$$

$$= -i C_F g^2 \int \frac{dk^+ dk^- d^2k_\perp}{(2\pi)^4} \frac{\bar{u}(p')\gamma^\alpha(\not{p}'-\not{k})\gamma_\mu(\not{p}-\not{k})\gamma_\alpha u(p)}{[2k^+k^- - k_\perp^2+i\varepsilon][-2(p^+-k^+)k^- + k_\perp^2 - i\varepsilon][2(p'^- - k^-)k^+ + k_\perp^2 - i\varepsilon]}$$

We now perform k^- integration with residues, which amounts in replacing

$$\frac{1}{2k^+k^- - k_\perp^2 + i\varepsilon} = \underbrace{(-2\pi i)}_{\text{Clockwise contour}} \delta(2k^+k^- - k_\perp^2) = \frac{-2\pi i}{2k^+}\,\delta\left(k^- - \frac{k_\perp^2}{2k^+}\right)$$

Neglecting when possible all terms of order $k_\perp$, we obtain

$$-i C_F g^2 \int \frac{dk^+ \cancel{dk^-} d^2k_\perp}{2k^+(2\pi)^3} \frac{\bar{u}(p')\gamma^\alpha(\not{p}'-\not{k})\gamma_\mu(\not{p}-\not{k})\gamma_\alpha u(p)}{\left[\frac{p^+}{k^+}k_\perp^2 - i\varepsilon\right]\left[2(p')^- k^+ - i\varepsilon\right]} =$$

↳ The component of p' is selected by k^+

$$= -i C_F g^2 \int \frac{dk^+}{k^+} \frac{1}{4p^+(p')^-} \cdot \int \frac{d^2k_\perp}{(2\pi)^3 k_\perp^2} \left[\bar{u}(p')\gamma^\alpha(\not{p}'-\not{k})\gamma_\mu(\not{p}-\not{k})\gamma_\alpha u(p)\right]$$

We now work out the numerator. If $\not{k} \simeq \xi\not{p}$ again $\not{k}u(p) \simeq 0$

$$\bar{u}(p')\gamma^\alpha(\not{p}'-\not{k})\gamma_\mu(\not{p}-\not{k})\gamma_\alpha u(p) = 2\,\bar{u}(p')(\not{p}-\not{k})(\not{p}'-\not{k})\gamma_\mu u(p) \simeq$$

$$\simeq 2\left(1-\frac{k^+}{p^+}\right)\bar{u}(p')\,\not{p}(\not{p}'-\not{k})\gamma_\mu u(p) \simeq 4p^+(p')^-\left(1-\frac{k^+}{p^+}\right)\bar{u}(p')\gamma_\mu u(p)$$

Since $\not{p}\not{k} = p^\mu\gamma_\mu k^\nu\gamma_\nu \simeq p^+k^+\gamma_+\gamma_+ = 0$ we have again factorisation of the hard vertex

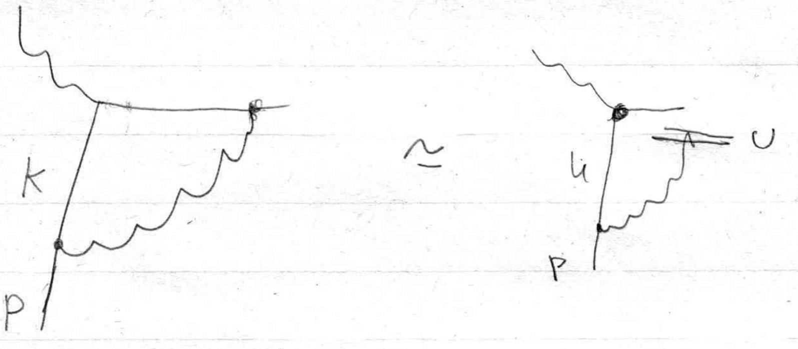
$$-i C_F \cdot g^2 \cdot \int \frac{dk^+}{k^+} \int \frac{d^2 k_\perp}{(2\pi)^3 k_\perp^2} \cdot \left(1 - \frac{k^+}{p^+}\right) \cdot \bar{u}(p') \gamma_\mu u(p)$$

We notice the fundamental properties:

1) Collinear singularities factor from the hard scattering

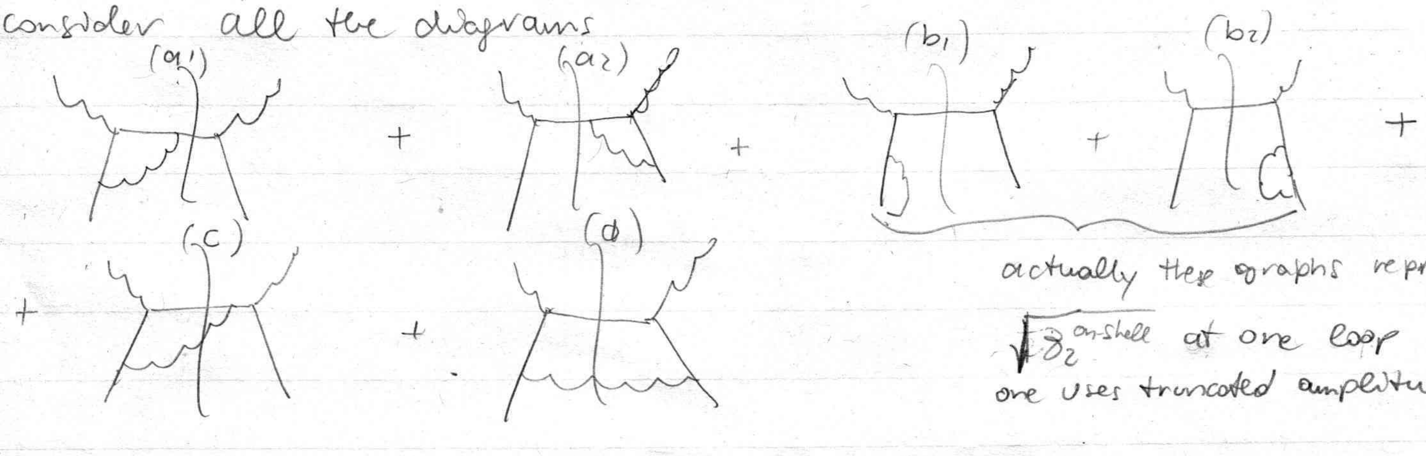
2) The propagator that does not contain the collinear singularity is large, e.g. $1/((p')^- k^+) \sim \frac{1}{Q^2}$

3) Gluons collinear to p couple to p' with eikonal Feynman rules but with $k \simeq k^+ \cdot v$ and $\varepsilon = \varepsilon^+ v$, which means that they couple effectively to a "fake" fermion moving with the velocity $U^\mu = g^\mu_-$ (this can be proven rigorously using Ward identities)



$(p')^- = (p' \cdot v) \cdot U^-$

If one considers the total DIS cross section one has to consider all the diagrams



actually these graphs represent $\sqrt{\delta_2^{\text{on-shell}}}$ at one loop if one uses truncated amplitudes

~~cut~~ final state cut $\frac{i}{k^2 + i\varepsilon} \to (2\pi)\, \delta(k^2)\, \Theta(k_0)$

1) All the previous graphs have collinear divergences. While ~~those~~ graphs (b) and (d) are naturally factorised from the hard scattering when the incoming line is proportional to p^+, this is not so evident for graphs (a) and (c).

Factorisation occurs only because gluons collinear to p ~~enter~~ the hard vertex with longitudinal momentum and polarisation, and due to a Ward identity, couple effectively to an eikonal line moving with velocity v such that $v \cdot v = 1$

In a phisical gauge, $n \cdot A = 0$, where

$$= + \frac{i}{k^2 + i\epsilon} \left[-g_{\mu\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{(n \cdot k)} + n^2 \frac{k_\mu k_\nu}{(n \cdot k)^2} \right]$$

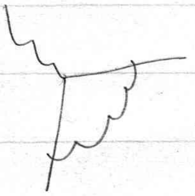
diagrams (a) and (c) have no collinear singularities, so that all collinear singularities are straightforwardly factorised.

2) Concerning soft singularities, those occurring in the final state cancel with virtual corrections (see exercise), while those occurring in the initial state might be again factored from the hard scattering. Notice that in this particular case, since we sum inclusively over all "hadronic" final states, and a soft gluon does not change the energy fraction entering the hard scattering, also initial state soft singularities cancel

3) Final state colinear singularities cancel completely with virtual corrections.

Singularities and pinch surfaces

Consider again our beloved vertex and Feynman parameterise it



$$\sim \int \frac{d^4k}{(2\pi)^4} \frac{N(p,p',k)}{[k^2+i\varepsilon][(p-k)^2+i\varepsilon][(p'-k)^2+i\varepsilon]} \approx$$

$$\sim \int \prod_{i=1}^{3} d\alpha_i \int \frac{d^4k}{(2\pi)^4} \frac{\delta(1-\sum_i \alpha_i)\, N(p,p',k)}{\left[\alpha_1(k^2+i\varepsilon)+\alpha_2[(p-k)^2+i\varepsilon)+\alpha_3[(p'-k)^2+i\varepsilon]\right]^3} \equiv D(\{\alpha_i\},k,p,p')$$

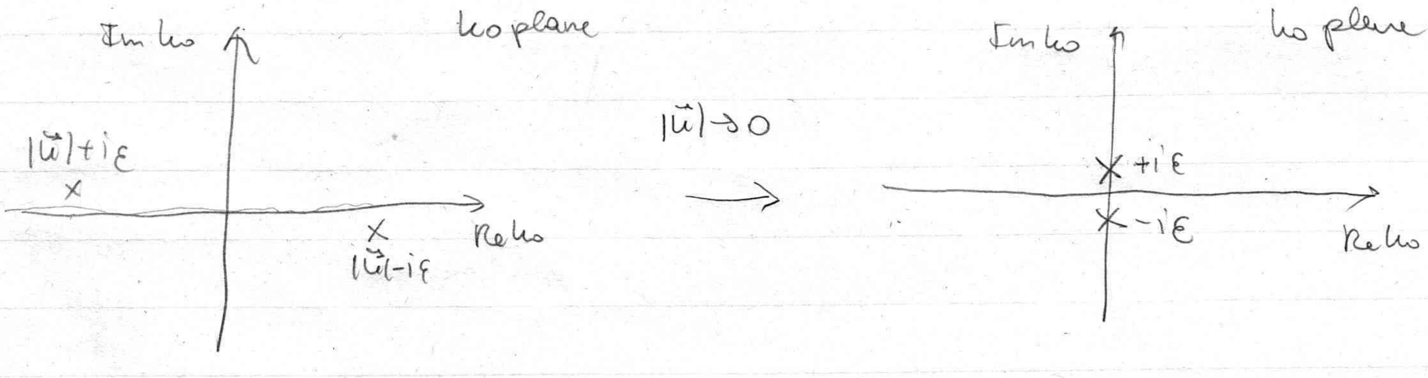
Singularities in the integral can occur of course only if the denominator vanishes. However, these are complex integral, so we can move the integration contour and avoid the singularity. This is however impossible in two cases

1) Endpoint singularity : the singularity occurs at the endpoint of one integration, the contour cannot be moved (e.g. any $\alpha_i = 0$)

2) Pinched singularity: the contour is pinched between two singularities

$$\times\ \bar{k}+i\varepsilon$$
$$\longrightarrow \ \mathrm{Re}\,k$$
$$\times\ \bar{k}-i\varepsilon$$

Example : soft singularity in the vertex case



Two solutions of $D(\{\alpha_i\}, k, p, p') = 0$ coincide only if

$$\frac{\partial}{\partial k^\mu} D(\{\alpha_i\}, k, p, p') = 0$$

We have then the two Landau equations which give necessary conditions for a singularity to occur

$$\begin{cases} D(\{\alpha_i\}, \{k_i\}, \{p\}) = 0 \\ \frac{\partial}{\partial k_i^\mu} D(\{\alpha_i\}, \{k_i\}, \{p\}) = 2 \sum_j \eta_{ij} \left[\alpha_j \cdot e_j^\mu\right] = 0 \end{cases}$$

where e_j^μ is the momentum of jth propagator

For instance for the one-loop vertex we have the following equations

$$\begin{cases} \alpha_1 (k^2 + i\epsilon) + \alpha_2 [(p-k)^2 + i\epsilon] + \alpha_3 [(p'-k)^2 + i\epsilon] = 0 \\ \alpha_1 k^\mu - \alpha_2 (p-k)^\mu - \alpha_3 (p'-k)^\mu = 0 \end{cases}$$

Which have the following sets of solutions

$$\begin{cases} k^\mu = \alpha_2 = \alpha_3 = 0 \\ (p-k)^\mu = \alpha_1 = \alpha_3 = 0 \\ (p'-k)^\mu = \alpha_1 = \alpha_2 = 0 \end{cases} \quad \text{soft}$$

$$\begin{cases} (p-k)^2 = k^2 = \alpha_3 = 0 \\ \alpha_1 k^\mu - \alpha_2 (p-k)^\mu = 0 \end{cases} \quad \text{collinear to } p$$

$$\begin{cases} (p'-k)^2 = k^2 = \alpha_2 = 0 \\ \alpha_1 k^\mu - \alpha_3 (p'-k)^\mu = 0 \end{cases} \quad \text{collinear to } p'$$