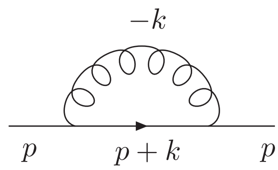
Introduction to QCD

FS 10, Series 7

Due date: 21.04.2010, 1 pm

Exercise 1

Consider the following (sub) diagram.

$M =$ 

This diagram was computed in Series 4 with mass m_0 and arbitrary incoming momentum p^2. For $m_0 = 0$ and $p^2 = 0$ show that the diagrams yields

$$\begin{aligned} M &= -ig^2\mu^{2\epsilon}C_F \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\mu(\not{p}+\not{k})\gamma_\mu}{[(p+k)^2][k^2]} \\ &= -ig^2\mu^{2\epsilon}C_F(d-2)\frac{\not{p}}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^4}. \end{aligned} \tag{1}$$

The Integral over k is scaleless, and should be zero in dimensional regularisation, this is due to the cancelation of IR and UV singularities. In what follows you will show that this is indeed the case. First change to Euclidean space by letting $k_0 \to ik_4$. Show that

$$I = \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^4} = -i\int \frac{d^dk_E}{(2\pi)^d}\frac{1}{k_E^4}.$$

Now change variables to spherical polar coordinates and split the integration region as follows

$$I = -i\int \frac{d\Omega^d}{(2\pi)^d}\left[\int_0^\Lambda dk_E k_E^{-1-2\epsilon} + \int_\Lambda^\infty dk_E k_E^{-1-2\epsilon}\right].$$

Finally show that

$$I = i\int d\Omega^d \left[\frac{\Lambda^{-2\epsilon_-}}{2\epsilon_-} - \frac{\Lambda^{-2\epsilon_+}}{2\epsilon_+}\right]$$

where $\epsilon_- < 0$ and $\epsilon_+ > 0$. Try to interprete your result.

Exercise 2

a) By integrating out p_2 and E_1 show that the d-dimensional $1 \to 2$ massless particle phasespace can be written as

$$\begin{aligned} d\Phi(\sqrt{s},0,0) &= \frac{d^{d-1}p_1}{(2\pi)^{d-1}}\frac{1}{2E_1}\frac{d^{d-1}p_2}{(2\pi)^{d-1}}\frac{1}{2E_2}(2\pi)^d\delta^{(d)}(q-p_1-p_2) \\ &= \frac{(4\pi)^{2\epsilon}}{2^5\pi^2}d\Omega_1^{3-2\epsilon}s^{-\epsilon} \\ &= \frac{s^{-\epsilon}(16\pi)^{\epsilon-1}}{\Gamma(1-\epsilon)}d\cos\theta(\sin\theta)^{-2\epsilon}. \end{aligned} \tag{2}$$

You should take $q = (\sqrt{s}, 0, .., 0)$. You may use that $d\Omega^d = d\cos\theta(\sin\theta)^{d-3}d\Omega^{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$.

b) Consider the $1 \to 3$ massless particle phasespace

$$d\Phi(\sqrt{s},0,0,0) = \frac{d^{d-1}p_1}{(2\pi)^{d-1}}\frac{1}{2E_1}\frac{d^{d-1}p_2}{(2\pi)^{d-1}}\frac{1}{2E_2}\frac{d^{d-1}p_3}{(2\pi)^{d-1}}\frac{1}{2E_3}(2\pi)^d\delta^{(d)}(q-p_1-p_2-p_3).$$

Rewrite the phasespace in terms of the energies E_1, E_2 and θ_{12} the (spacial) angle between particle 1 and 2. Hence confirm that

$$\begin{aligned} d\Phi(\sqrt{s},0,0,0) &= \frac{(2\pi)}{4}^{3-2d} d\Omega_1^{d-2}d\Omega_2^{d-1}dE_1dE_2(E_1E_2)^{d-3}d\cos\theta_{12}(\sin\theta_{12})^{d-4} \\ &\quad \delta(s-2\sqrt{s}(E_1+E_2)+2E_1E_2(1-\cos(\theta_{12}))). \end{aligned} \tag{3}$$

Now change variables to $x_i = \frac{2p_i.q}{s}$ for $i = 1, 2, 3$ to derive

$$\begin{aligned} \Phi(\sqrt{s},0,0,0) &= s^{1-2\epsilon}2^{-10+6\epsilon}\pi^{-5+4\epsilon}\int d\Omega_1^{2-2\epsilon}d\Omega_2^{3-2\epsilon} \\ &\quad \int_0^1 dx_1dx_2dx_3\,[x_1(1-x_1)x_2(1-x_2)(1-x_3)]^{-\epsilon} \\ &\quad \delta(2-x_1-x_2-x_3). \end{aligned} \tag{4}$$

Exercise 3 The aim of this exercise is to compute the leading order cross section of the process $e^+e^- \rightarrow Hadrons$ in $d = 4 - 2\epsilon$ dimensions. Where by *Hadrons* we really mean all (here massless) $q\bar{q}$ pairs whose center of mass energy does not exceed the collision energy. Why is that sufficient?
The compution is then up to a factor of $N_c(\sum Q_f^2)$ (why?) completely analagous to the case of $e^+e^- \rightarrow \mu^+\mu^-$, which we computed in **Series 1**. Where Q_f shall denote the fractional electric charge of the quark f. The amplitude squared is thus just

$$|M(e^+e^- \rightarrow Hadrons)|^2 = e^4 N_c(\sum_f = 1_f^n Q_f^2)\left(\frac{L_e^{\mu\nu} L_{\mu\nu}^{\texttt{hadron}}}{s^2}\right)$$

where

$$\begin{aligned} L_e^{\mu\nu} &= (p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - p_1.p_2 g^{\mu\nu}) \\ L_{\texttt{muon}}^{\mu\nu} &= 4\,(p_3^\mu p_4^\nu + p_3^\nu p_4^\mu - p_3.p_4 g^{\mu\nu})\,. \end{aligned} \tag{5}$$

a) Show that in $d = 4 - 2\epsilon$ dimensions

$$|M(e^+e^- \rightarrow Hadrons)|^2 \;=\; 2e^4 N_c\left(\sum_{f=1}^{n_f} Q_f^2\right)\left(\frac{u^2+t^2}{s^2} - \epsilon\right). \tag{6}$$

b) Use the d-dimensional phase space computed in Exercise 2.a) to derive the d-dimensional leading order cross section.

$$\sigma_{e^+e^- \rightarrow Hadrons} = \frac{4\pi\alpha_s^2}{3s}\left(\sum_{f=1}^{n_f} Q_f^2\right)\left(\frac{4\pi}{s}\right)^\epsilon \frac{3(1-\epsilon)\Gamma(2-\epsilon)}{(3-2\epsilon)\Gamma(2-2\epsilon)}$$