Exercise 1

Consider the following (sub) diagram.

\[ M = \begin{array}{c}
\text{\( -k \)} \\
p \\
\text{\( p + k \)} \\
p
\end{array} \]

This diagram was computed in Series 4 with mass \( m_0 \) and arbitrary incoming momentum \( p^2 \). For \( m_0 = 0 \) and \( p^2 = 0 \) show that the diagrams yields

\[
M = -ig^2 \mu^{2\epsilon} C_F \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\mu(p + k)\gamma_\mu}{[(p + k)^2][k^2]}
\]

\[
= -ig^2 \mu^{2\epsilon} C_F (d - 2) \frac{d^\epsilon}{2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^4}.
\]  

(1)

The integral over \( k \) is scaleless, and should be zero in dimensional regularisation, this is due to the cancelation of IR and UV singularities. In what follows you will show that this is indeed the case. First change to Euclidean space by letting \( k_0 \to ik_4 \). Show that

\[
I = \int \frac{d^d\epsilon}{(2\pi)^d} \frac{1}{k^4} = -i \int \frac{d^d\epsilon}{(2\pi)^d} \frac{1}{k^4_E}.
\]

Now change variables to spherical polar coordinates and split the integration region as follows

\[
I = -i \int \frac{d\Omega^d}{(2\pi)^d} \left[ \int_0^\Lambda dk_E k_E^{-1-2\epsilon} + \int_\Lambda^\infty dk_E k_E^{-1-2\epsilon} \right].
\]

Finally show that

\[
I = i \int d\Omega^d \left[ \frac{\Lambda_{-2\epsilon}}{2\epsilon_-} - \frac{\Lambda_{-2\epsilon_+}}{2\epsilon_+} \right]
\]

where \( \epsilon_- < 0 \) and \( \epsilon_+ > 0 \). Try to interpret your result.
Exercise 2

a) By integrating out $p_2$ and $E_1$ show that the d-dimensional $1 \rightarrow 2$ massless particle phasespace can be written as

$$d\Phi(\sqrt{s}, 0, 0) = \frac{d^{d-1}p_1}{(2\pi)^{d-1}} \frac{1}{2E_1} \frac{d^{d-1}p_2}{(2\pi)^{d-1}} \frac{1}{2E_2} (2\pi)^d \delta^{(d)}(q - p_1 - p_2)$$

$$= \frac{(4\pi)^{2\epsilon}}{2^{5\epsilon} \pi^2} d\Omega_1^{3-2\epsilon} s^{-\epsilon}$$

$$= \frac{s^{-\epsilon}(16\pi)^{\epsilon-1}}{\Gamma(1-\epsilon)} d\cos(\theta(\sin \theta)^{-2\epsilon}).$$

(2)

You should take $q = (\sqrt{s}, 0, .., 0)$. You may use that $d\Omega^d = d\cos(\theta(\sin \theta) d\Omega^{d-1} = \frac{2\pi}{\Gamma(\frac{d}{2})}$.  

b) Consider the $1 \rightarrow 3$ massless particle phasespace

$$d\Phi(\sqrt{s}, 0, 0, 0) = \frac{d^{d-1}p_1}{(2\pi)^{d-1}} \frac{1}{2E_1} \frac{d^{d-1}p_2}{(2\pi)^{d-1}} \frac{1}{2E_2} \frac{d^{d-1}p_3}{(2\pi)^{d-1}} \frac{1}{2E_3} (2\pi)^d \delta^{(d)}(q - p_1 - p_2 - p_3).$$

Rewrite the phasespace in terms of the energies $E_1$, $E_2$ and $\theta_{12}$ the (spacial) angle between particle 1 and 2. Hence confirm that

$$d\Phi(\sqrt{s}, 0, 0, 0) = \frac{(2\pi)^{3-2d}}{4} d\Omega_1^{d-2} d\Omega_2^{d-1} dE_1 dE_2 (E_1 E_2)^{d-3} d\cos(\theta_{12}) (\sin \theta_{12})^{d-4}$$

$$\delta(s - 2\sqrt{s}(E_1 + E_2) + 2E_1 E_2(1 - \cos(\theta_{12}))).$$

(3)

Now change variables to $x_i = \frac{2E_i q}{s}$ for $i = 1, 2, 3$ to derive

$$\Phi(\sqrt{s}, 0, 0, 0) = s^{1-2\epsilon} 2^{-10+6\epsilon} \pi^{-5+4\epsilon} \int d\Omega_1^{2-2\epsilon} d\Omega_2^{3-2\epsilon}$$

$$\int_0^1 dx_1 dx_2 dx_3 [x_1(1-x_1)x_2(1-x_2)(1-x_3)]^{-\epsilon}$$

$$\delta(2 - x_1 - x_2 - x_3).$$

(4)
Exercise 3  The aim of this exercise is to compute the leading order cross section of the process $e^+e^- \rightarrow \text{Hadrons}$ in $d = 4 - 2\epsilon$ dimensions. Where by Hadrons we really mean all (here massless) $q\bar{q}$ pairs whose center of mass energy does not exceed the collision energy.

Why is that sufficient?

The computation is then up to a factor of $N_c(\sum Q_f^2)$ (why?) completely analoguous to the case of $e^+e^- \rightarrow \mu^+\mu^-$, which we computed in Series 1. Where $Q_f$ shall denote the fractional electric charge of the quark $f$. The amplitude squared is thus just

$$|M(e^+e^- \rightarrow \text{Hadrons})|^2 = e^4 N_c(\sum_{j=1}^{n_f} 1^j Q_f^2) \left( \frac{L^{\mu\nu}_{\text{hadron}}}{s^2} \right)$$

where

$$L^{\mu\nu}_e = \left( p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - p_1 p_2 g^{\mu\nu} \right)$$

$$L^{\mu\nu}_{\mu\text{on}} = 4 \left( p_3^\mu p_4^\nu + p_3^\nu p_4^\mu - p_3 p_4 g^{\mu\nu} \right).$$

(5)

a) Show that in $d = 4 - 2\epsilon$ dimensions

$$|M(e^+e^- \rightarrow \text{Hadrons})|^2 = 2 e^4 N_c \left( \sum_{j=1}^{n_f} Q_f^2 \right) \left( \frac{u^2 + t^2}{s^2} - \epsilon \right).$$

(6)

b) Use the d-dimensional phase space computed in Exercise 2.a) to derive the d-dimensional leading order cross section.

$$\sigma_{e^+e^-\rightarrow\text{Hadrons}} = \frac{4\pi\alpha_s^2}{3s} \left( \sum_{j=1}^{n_f} Q_f^2 \right) \left( \frac{4\pi}{s} \right)^\epsilon \frac{3(1 - \epsilon)\Gamma(2 - \epsilon)}{(3 - 2\epsilon)\Gamma(2 - 2\epsilon)}$$