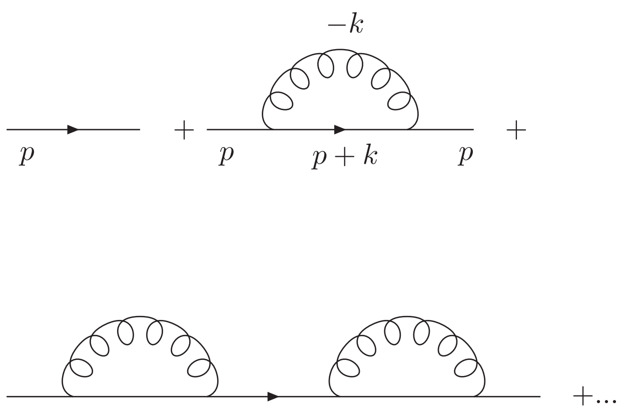
Introduction to QCD

FS 10, Series 4

Due date: 24.03.2010, 1 pm

Exercise 1 Consider inserting 1 particle irreducible (1PI) diagrams into the quark propagator as illustrated in the following diagram, here only at 2nd order in perturbation theory.

$S_Q(p) =$



a) Let $-i\Sigma$ denote the sum of all those 1PI diagrams. Review that we can then resum the exact propagator using the geometric series to give

$$S_Q(p) = \frac{i}{\not{p} - m_0 - \Sigma(p, m_0)}.$$

where m_0 shall denote the bare mass.

b) Show that at the 2nd order in perturbation theory in $d = 4 - 2\epsilon$ dimensions

$$\begin{aligned}
\Sigma(p) &= -ig^2\mu^{2\epsilon}C_F \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\mu(\not{p} + \not{k} + m_0)\gamma_\mu}{[(p+k)^2 - m_0^2][k^2]}. \\
&= -ig^2\mu^{2\epsilon}C_F \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{dm_0 + (2-d)\not{p}(1-x)}{[k^2 - (m_0^2 - p^2(1-x))x]^2} \\
&= A(p^2, m_0^2)m_0 + B(p^2, m_0^2)\not{p}
\end{aligned} \tag{1}$$

where

$$\begin{aligned}
A(p^2, m_0^2) &= \frac{\alpha_s}{4\pi}\mu^{2\epsilon}C_F(4\pi)^\epsilon\Gamma(\epsilon)(4-2\epsilon)\int_0^1 dx(m_0^2x - p^2x(1-x))^{-\epsilon} \\
B(p^2, m_0^2) &= \frac{\alpha_s}{4\pi}\mu^{2\epsilon}C_F(4\pi)^\epsilon\Gamma(\epsilon)(2\epsilon-2)\int_0^1 dx(m_0^2x - p^2x(1-x))^{-\epsilon}(1-x).
\end{aligned} \tag{2}$$

c) Show that to $O(\epsilon)$ the expressions for A and B become

$$\begin{aligned} A(p^2, m_0^2) &= \frac{\alpha_s}{4\pi} C_F \left(\frac{4}{\epsilon} - 4\gamma_E + 4\log(4\pi) - 2 - 4\int_0^1 dx \log(\frac{m_0^2 x - p^2 x(1-x)}{\mu^2}) \right) \\ B(p^2, m_0^2) &= \frac{\alpha_s}{4\pi} C_F \left(-\frac{1}{\epsilon} + \gamma_E - \log(4\pi) + 1 + 2\int_0^1 dx (1-x) \log(\frac{m_0^2 x - p^2 x(1-x)}{\mu^2}) \right) \end{aligned} \tag{3}$$

Exercise 2 The ultraviolet divergences can be renormalised into wave function and mass by defining

$$\begin{aligned} \psi_0 &= \sqrt{Z_2}\psi \\ m_0 &= Z_m m, \end{aligned} \tag{4}$$

such that Z_m and Z_2 now contain the ultraviolet divergences. Show that in terms of the renormalised quantities the propagator becomes

$$S_Q(p) = \frac{i}{Z_2 \not{p} - Z_2 Z_m m - \Sigma(p, m)}.$$

Deduce then that in general we must have

$$\begin{aligned} Z_2 &= 1 - \frac{\alpha_s}{4\pi} C_F \left(\frac{1}{\epsilon} + \textbf{finite} \right) \\ Z_m &= 1 - \frac{\alpha_s}{4\pi} 3C_F \left(\frac{1}{\epsilon} + \textbf{finite} \right). \end{aligned} \tag{5}$$

(i) The Minimal Subtraction scheme (MS) is defined such that only the ultraviolet divergences are absorbed into the Z's, the **finite** piece is then dropped.

(ii) In the modified minmal subtraction ($\overline{MS}$) scheme one lets **finite** $= -\gamma_E + \log(4\pi)$.

(iii) In the onshell scheme one defines the Z's in such a way to guarantee that the pole of $S_Q(p)$ occurs at $\not{p} = m$ with residue 1. This is guaranteed by the following conditions.

$$\begin{aligned} (Z_2\not{p} - Z_2 Z_m m - \Sigma(p, m))\,|_{\not{p}\,=\,m} &= 0 \\ \frac{d}{d\not{p}}(Z_2\not{p} - Z_2 Z_m m - \Sigma(p, m))\,|_{\not{p}\,=\,m} &= 1 \end{aligned} \tag{6}$$