Exercise 1

a) Consider electron positron annihilation into photons in QED, i.e. \( e^+e^- \rightarrow \gamma\gamma \). At leading order the Feynman diagrams contributing to this process are

\[
\begin{align*}
M^{\mu\nu}_{e^+e^-\rightarrow\gamma\gamma} &= \gamma, p_4, \nu & e^+, p_2 = M^{\mu\nu}_{1} \gamma, p_3, \mu \\
&= M^{\mu\nu}_{2} \gamma, p_4, \nu & e^+, p_2 = M^{\mu\nu}_{1} \gamma, p_3, \mu
\end{align*}
\]

Show that

\[
M^{\mu\nu}_{1} = (-ie^2) \bar{v}_2 \gamma^\mu (p_3 - p_2) \gamma^\nu u_1 \frac{1}{(p_2 - p_3)^2}
\]

\[
M^{\mu\nu}_{2} = (-ie^2) \bar{v}_2 \gamma^\nu (p_4 - p_3) \gamma^\mu u_1 \frac{1}{(p_1 - p_3)^2}
\]

(1)

b) Verify that the QED Ward identity

\[
(p_3)_\mu M^{\mu\nu}_{e^+e^-\rightarrow\gamma\gamma} = 0 = (p_4)_\mu M^{\mu\nu}_{e^+e^-\rightarrow\gamma\gamma}
\]

is fulfilled. Conclude that QED amplitudes are purely transverse.

Exercise 2

a) Now consider the process \( q\bar{q} \rightarrow gg \) in QCD. At leading order the following diagrams contribute
\[ M_{q\bar{q} \rightarrow gg}^{\mu\nu} = \]
\[ = M_1^{\mu\nu} + M_2^{\mu\nu} = M_3^{\mu\nu} \]
\[ \text{and} \]
\[ M_{\text{Ghost}}^{\mu\nu} \]

Show that the above diagrams yield
\[ M_1^{\mu\nu} = (-ig^2)(T^a T^b)_{j_1} \bar{v}_2 \gamma^\mu (p_3 - p_2) \gamma^\nu u_1 \]
\[ M_2^{\mu\nu} = (-ig^2)(T^b T^a)_{j_3} \bar{v}_2 \gamma^\nu (p_1 - p_3) \gamma^\mu u_1 \]
\[ M_3^{\mu\nu} = (-ig^2)[T^a, T^b]_{j_3} \bar{v}_2 p_3 u_1 - (p_3 + 2p_4)\gamma^\nu \bar{v}_2 \gamma^\mu u_1 + (p_4 + 2p_3)\gamma^\nu \bar{v}_2 \gamma^\mu u_1 \]
\[ M_{\text{Ghost}}^{\mu\nu} = (-ig^2)[T^a, T^b]_{j_1} \bar{v}_2 p_3 u_1 \]

(2)

b) Confirm that the QCD Ward identity is fulfilled
\[ (p_3)_\mu M_{q\bar{q} \rightarrow gg}^{\mu\nu} = p_4^\nu M_{\text{Ghost}}. \]

**Exercise 3** Consider soft radiative QCD corrections to a quark scattering off an electromagnetic current.
a) Show that (in the soft limit) the matrix element becomes

\[ M^{\mu\alpha}_{ij} = (-ie\bar{u}_2\gamma^{\mu}u_1) J^{\alpha a}_{ij} (1 + O(k)) \]

where

\[ J^{\alpha a}_{ij} = -g(T^a)_{ji} \left[ \frac{p_2^2}{p_{2.k}} - \frac{p_2^2}{p_{1.k}} \right]. \]

Hence prove that \( J^{\alpha a}_{ij} k_\alpha = 0 \) to show that the current \( J \) is conserved.

b) Compute the square of the matrix element, i.e.

\[ |M^{\mu\nu}|^2 = \sum_{\lambda} \epsilon_\alpha(\lambda) (\epsilon^*)_\beta(\lambda) M^{\mu\alpha a}_{ij} (M^{\ast})_{ij}^{\nu3a} \]

and show that the matrix element square factorises into a soft (eikonal) part times the squared amplitude of the underlying leading order process \(|M_0^{\mu\nu}|^2\) as follows

\[ |M^{\mu\nu}|^2 = -g^2C_F \left( \frac{p_1^2}{p_{1.k}} - \frac{2p_1.p_2}{p_{1.k}p_{2.k}} + \frac{p_2^2}{p_{2.k}} \right) |M_0^{\mu\nu}|^2. \]