# Introduction to QCD

FS 10, Series 5

Due date: 12.04.2010, 6 pm

---

**Exercise 1**

Consider the group $SU(N)$ of the unitary $N \times N$ matrices $U$ with unit determinant

$$UU^\dagger = U^\dagger U = \mathbb{1}, \qquad \det U = 1\,. \tag{1}$$

Show that these constraints imply that the *real* dimension of the group is $N^2 - 1$.

**Exercise 2**

Consider a vector $q$ belonging to the *fundamental* representation of $SU(N)$:

$$q^i \to (q^i)' = U^i_j\, q^j\,, \qquad U = e^{-i\theta_a t^a}\,, \quad a = 1, \ldots N^2 - 1\,. \tag{2}$$

(i) Show that the generators $t^a$ are hermitian traceless matrices.

(ii) The vector $q^\dagger$ belongs to the *conjugate* representation. Show that the generators of the conjugate representation are $\bar{t}^a = -(t^a)^T$.

**Exercise 3**

The generators $T^a(R)$ of any representation $R$ obey the commutation rules

$$[T^a(R), T^b(R)] = i f^{abc}\, T^c(R)\,. \tag{3}$$

Show that the generators $T^a$ of the adjoint representation, given by $(T^a)_{bc} = -if_{abc}$, satisfy this commutation relations.
*Hint:* Use the Jacobi Identity ( $[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0$ ).

**Exercise 4**

For each representation, consider the *quadratic Casimir* operator $T^2(R) = T^a(R) T^a(R)$.

(i) Show that $T^2$ commutes with every generator $T^a$, and hence $T^2(R) = C_R\, \mathbb{1}_R$, where $\mathbb{1}_R$ is the identity matrix in the vector space spanning representation $R$.

(ii) Given that the generators of each representation are normalised as follows

$$\mathrm{Tr}[T_a(R)\, T_b(R)] = T_R\, \delta_{ab}\,. \tag{4}$$

where $T_R$ depends on the representation, show that $T_R$ and $C_R$ are related by

$$C_R\ \dim(R) = T_R\ \dim(G)\,. \tag{5}$$

(iii) Given the normalisation $T_F = 1/2$, derive

$$C_F = \frac{N^2-1}{2N} \qquad C_A = T_A = N\,. \tag{6}$$

*Hint*: To calculate $T_A$ it is helpful to use the Fierz identity: $t^a_{ij} t^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{kl} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)$

### Exercise 5

Using the fact that a QCD amplitude can be written as the product of a purely kinematical (Lorentz invariant) part times a colour part, calculate the colour factors that appear in the following squared matrix elements at leading order: $|\mathcal{M}(qq' \longrightarrow qq')|^2$, $|\mathcal{M}(q\bar{q} \longrightarrow q'\bar{q'})|^2$, $|\mathcal{M}(qg \longrightarrow qg)|^2$, $|\mathcal{M}(q\bar{q} \longrightarrow gg)|^2$.