Exercise 1

Consider the group $SU(N)$ of the unitary $N \times N$ matrices $U$ with unit determinant

$$UU^\dagger = U^\dagger U = 1, \quad \det U = 1.$$  \hspace{1cm} (1)

Show that these constraints imply that the real dimension of the group is $N^2 - 1$.

Exercise 2

Consider a vector $q$ belonging to the fundamental representation of $SU(N)$:

$$q^i \rightarrow (q^i)' = U^i_j q^j, \quad U = e^{-i\theta_a t^a}, \quad a = 1, \ldots N^2 - 1.$$ \hspace{1cm} (2)

(i) Show that the generators $t^a$ are hermitian traceless matrices.

(ii) The vector $q^\dagger$ belongs to the conjugate representation. Show that the generators of the conjugate representation are $\bar{t}^a = -(t^a)^T$.

Exercise 3

The generators $T^a(R)$ of any representation $R$ obey the commutation rules

$$[T^a(R), T^b(R)] = i f^{abc} T^c(R).$$ \hspace{1cm} (3)

Show that the generators $T^a$ of the adjoint representation, given by $(T^a)_{bc} = -if_{abc}$, satisfy this commutation relations.

Hint: Use the Jacobi Identity ($[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0$).

Exercise 4

For each representation, consider the quadratic Casimir operator $T^2(R) = T^a(R)T^a(R)$.

(i) Show that $T^2$ commutes with every generator $T^a$, and hence $T^2(R) = C_R \mathbb{1}_R$, where $\mathbb{1}_R$ is the identity matrix in the vector space spanning representation $R$. 
(ii) Given that the generators of each representation are normalised as follows
\[ \text{Tr}[T_a(R) T_b(R)] = T_R \delta_{ab}. \] 
where \( T_R \) depends on the representation, show that \( T_R \) and \( C_R \) are related by
\[ C_R \dim(R) = T_R \dim(G). \] 

(iii) Given the normalisation \( T_F = 1/2 \), derive
\[ C_F = \frac{N^2 - 1}{2N}, \quad C_A = T_A = N. \] 

*Hint:* To calculate \( T_A \) it is helpful to use the Fierz identity:
\[ \epsilon_{ijkl} \epsilon_{ij'mn} = \frac{1}{2} \left( \delta_{i'm} \delta_{jl} - \frac{1}{N} \delta_{ij} \delta_{kl} \right). \]

**Exercise 5**

Using the fact that a QCD amplitude can be written as the product of a purely kinematical (Lorentz invariant) part times a colour part, calculate the colour factors that appear in the following squared matrix elements at leading order: \(|M(qq' \rightarrow qq')|^2, |M(q\bar{q} \rightarrow q'\bar{q}')|^2, |M(qg \rightarrow qg)|^2, |M(q\bar{q} \rightarrow gg)|^2, |M(qg \rightarrow qg)|^2, |M(q\bar{q} \rightarrow gg)|^2.\)