Exercise 1: one-loop thrust distribution

1. Recall the definition of the thrust in $e^+e^-$ annihilation:

\[ T = \max_{\vec{n}} \frac{\sum_i |\vec{q}_i \cdot \vec{n}|}{\sum_i |\vec{q}_i|}. \quad (1) \]

The axis $\vec{n}_T$ that maximises the sum eq. (1) is called the thrust axis.

(a) Show that for two (back-to-back) massless particles, the thrust axis is along the direction of any of the two. Show also that in this case $T = 1$.

(b) Optional: show that for three massless particles \( \{q_1, q_2, q_3\} \) the thrust axis is along the most energetic particle $q_1$. Using also $q = q_1 + q_2 + q_3$ with $q^2 = Q^2$ show that

\[ T = \frac{2(q_1q)}{Q^2} = 1 - \frac{(q_2 + q_3)^2}{Q^2}. \quad (2) \]

Hint: use the fact that the thrust axis divides the event in two hemispheres (L and R), such that

\[ \sum_{i \in R} \vec{q}_i \times \vec{n}_T = \sum_{i \in L} \vec{q}_i \times \vec{n}_T = 0. \quad (3) \]

2. Consider the one-loop thrust distribution in $e^+e^-$ annihilation:

\[ \Sigma(\tau) = 1 + 2C_F g^2 \int [dk] \frac{p\bar{p}}{(pk)(k\bar{p})} \left[ \Theta \left( \tau - \frac{\omega}{Q}(1 - |\cos \theta|) \right) - 1 \right] \quad (4) \]

where momenta, neglecting recoil, are parametrised as follows

\[ p = \frac{Q}{2} (1, 0, 0, 1), \quad \bar{p} = \frac{Q}{2} (1, 0, 0, -1), \quad k = \omega(1, 0, \sin \theta, \cos \theta). \quad (5) \]

After having verified that

\[ [dk] = \frac{\omega d\omega}{8\pi^2} d\cos \theta \frac{d\phi}{2\pi}, \quad \frac{p\bar{p}}{(pk)(k\bar{p})} = \frac{2}{\omega^2(1 - \cos^2 \theta)}, \quad (6) \]

using $g^2 = 4\pi\alpha_s$, work out the integration in eq. (4) and show that

\[ \Sigma(\tau) = 1 - C_F \frac{\alpha_s}{\pi} \left[ \ln^2 \frac{1}{\tau} + \mathcal{O}(1) \right]. \quad (7) \]
Exercise 2: two-jet rate

A clustering algorithm in $e^+e^-$ annihilation is a recursive procedure that works as follows. Fix a minimum required number of jets $n_{\text{jets}}$ and let $n$ be the number of particles.

1. If $n = n_{\text{jets}}$ stop: the number of particles $n$ is equal to the required number of jets $n_{\text{jets}}$. If $n > n_{\text{min}}$, for any pair of particles $q_i,q_j$, construct an IRC safe distance $d_{ij}$, for instance

   \[d_{ij} = \begin{cases} 
   2E_iE_j(1 - \cos \theta_{ij}) & \text{(JADE)} \\
   2 \min\{E_i^2, E_j^2\}(1 - \cos \theta_{ij}) & \text{(Durham)}
   \end{cases} \quad (8)\]

2. Merge the pair $q_i,q_j$ for which $d_{ij}$ is minimum and replace them with a pseudoparticle $q_{ij}$, for instance $q_{ij} = q_i + q_j$ (E-scheme). Define also $d_n = d_{ij}$

3. Let $n \to n - 1$ and go back to step 1.

Let us define the $n$-jet resolution $y_{n-1,n} \equiv d_n/Q^2$. The $n$-jet rate $R_n(y_{\text{cut}})$ is defined as the fraction of events for which $y_{n,n+1} < y_{\text{cut}} < y_{n-1,n}$, with $y_{\text{cut}}$ a given resolution parameter. It is evident that $R_2(y_{\text{cut}})$ is the fraction of events for which $y_{23} < y_{\text{cut}}$.

For $n \leq 3$ show that

\[R_2(y_{\text{cut}}) = 1 - R_3(y_{\text{cut}}) \simeq 1 - 2C_F g^2 \int [dk] \frac{p\bar{p}}{(pk)(k\bar{p})} \Theta (d_3(k) - y_{\text{cut}}Q^2) , \quad (9)\]

where the last approximation holds in the soft-collinear limit.

Using the kinematics of eq. (5), work out the integral in eq. (9) for both the JADE and the Durham algorithm and show that

\[R_2(y_{\text{cut}}) \simeq \begin{cases} 
1 - C_F \frac{\alpha_s}{\pi} \ln^2 \frac{1}{y_{\text{cut}}} & \text{(JADE)} \\
1 - \frac{C_F}{2\pi} \ln^2 \frac{1}{y_{\text{cut}}} & \text{(Durham)}
\end{cases} \quad (10)\]

where in the above expression we have neglected terms that do not contain logarithms of the jet-resolution parameter $y_{\text{cut}}$. 

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