LECTURE 9

Monopoles

We saw, with solitons, how classical solutions of the eq. of motion can interpolate non-trivially between the discrete zero's of the potential of the theory.

The next most complicated case is when the potential vanishes in a continuous manifold. ~~We will study~~ This involves spontaneous symmetry breaking.

The Georgi-Glasow model

$$\mathcal{L} = -\frac{1}{4} F^i_{\mu\nu} F^{\mu\nu i} - \frac{1}{2} D_\mu \vec{\phi}\, D^\mu \vec{\phi} - V(\vec{\phi}^{\,2})$$

$\vec{\phi} = (\phi_1, \phi_2, \phi_3)$, and the gauge group is $SU(2)$

so
$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + e\, \varepsilon^{ijk} A^j_\mu A^k_\nu$$

$$(D_\mu \vec{\phi})_i = \partial_\mu \phi_i + e\, A^j_\mu\, \varepsilon_{ijk}\, \phi_k$$

Also,
$$V(\vec{\phi}) = \lambda \left(\vec{\phi}^{\,2} - \langle\phi\rangle^2 \right)^2$$

Trivial solution that minimizes the potential

$$A^i_\mu = 0 \qquad \phi^*_i : \ \vec{\phi}^{*2} = \langle\phi\rangle^2$$

Note that SSB is happening here: ~~setting~~ assume $\vec{\phi}^* = \begin{pmatrix} 0 \\ 0 \\ \langle\phi\rangle \end{pmatrix}$

and set $\phi_3 = \phi^*_3 + \phi'_3 = \langle\phi\rangle + \phi'_3$ in $\mathcal{L}$. Then there is a term

$$D_\mu \vec{\phi}\, D^\mu \vec{\phi} = \ldots + e^2 \varepsilon_{ijk} A^j_\mu \phi_k\, \varepsilon_{imn} A^m_\mu \phi_n$$
$$= e^2 \left(A^j_\mu \phi_k A^j_\mu \phi_k - A^j_\mu \phi_k A^k_\mu \phi_j \right) + \ldots$$
$$= \ldots + e^2 \left(A^j_\mu A^j_\mu \langle\phi\rangle^2 - A^3_\mu A^3_\mu \langle\phi\rangle^2 \right)$$
$$= \ldots + e^2 \langle\phi\rangle^2 \left(A^1_\mu A^1_\mu + A^2_\mu A^2_\mu \right)$$

hence $SU(2) \rightarrow U(1)$, the fields A_μ^1, A_μ^2 acquire mass, and A_μ^3 remains a massless field which is identified with the photon.

Non-trivial solutions

Are there non-trivial solutions in the temporal gauge, $A_0^i = 0$, time-independent $\left(\partial_0 A_i^m = 0 \right.$, $\left. \partial_0 \vec{\phi} = 0 \right)$?

Then $F_{\mu\nu}^i F^{i\mu\nu} = F_{jk}^i F^{ijk}$ $\quad D_\mu \vec{\phi} D^\mu \vec{\phi} = D_i \vec{\phi} D^i \vec{\phi}$

so $\mathcal{L} = -\mathcal{H}$ with $\mathcal{H} = \frac{1}{4} F_{jk}^i F^{ijk} + \frac{1}{2} D_i \vec{\phi} D_i \vec{\phi} + V(\vec{\phi})$

In order for the integral to be finite, each term has to be finite

$$\int d^4x \, V(\phi) < \infty \quad \Rightarrow \quad \lim_{x \to \infty} \vec{\phi}(\vec{x}) = \vec{\phi}^*(x) \quad \text{with } |\vec{\phi}^*|^2 = \langle \phi \rangle^2$$

But where on the sphere should $\vec{\phi}(\vec{x})$ tend assymptotically?

Once again, the solutions are classified by homotopy classes and winding numbers.

The e/m field tensor ('t Hooft)

In the trivial solution we saw that the direction of $\vec{\phi}^*$ ~~decide~~ determines the direction of the unbroken $U(1)$ that is the photon.

Now, the direction of $\vec{\phi}^*$ changes with the spatial direction.

Let's define $F'_{\mu\nu} = F_{\mu\nu}^i \hat{\phi}_i - \frac{1}{e} \varepsilon_{ijk} \hat{\phi}_i D_\mu \hat{\phi}_j D_\nu \hat{\phi}_k$

$$\hat{\phi}_i = \frac{\phi_i}{\sqrt{\phi_j \phi_j}}$$

Then $F'_{\mu\nu} = \partial_\mu \left(\hat{\phi}_i A_\nu^i \right) - \partial_\nu \left(\hat{\phi}_j A_\mu^i \right) - \frac{1}{e} \varepsilon_{ijk} \hat{\phi}_i \partial_\mu \hat{\phi}_j \partial_\nu \hat{\phi}_k$

If we can find a gauge ($SU(2)$ in the adjoint) trs. to make the field $\vec{\phi}$ point at a certain direction, say the z-axis, in a region in space, then $F'_{\mu\nu} = \partial_\mu(\hat{\varphi}_i A^i_\nu) - \partial_\nu(\hat{\varphi}_i A^i_\mu)$ and it's that direction of A^i_μ, in that region, that remains unbroken. So it is reasonable to define $F'_{\mu\nu}$ to be the $U(1)$ e/m tensor

* There is always such a gauge trs. if we restrict ourselves to half the space, say when $0 < \theta < \pi$. There is another one when $\pi < \theta < 2\pi$. The boundary conditions are related to the quantization of the monopole charge a la Dirac (see Weinberg p 441, 442).

The magnetic monopole moment

The electric charge (aka the electric monopole moment) is the source of the electric field, as seen by Gauss' law

$$q = \frac{1}{4\pi} \oint_S \vec{E} \cdot d\vec{s}$$

Similarly the magnetic monopole moment (the magnetic charge) is defined as
$$g = \frac{1}{4\pi} \oint_S \vec{B} \cdot d\vec{s} = \frac{1}{4\pi} \frac{1}{2} \varepsilon_{ijk} \oint_S F'_{ij} \, d^2 S_k$$

(i) g is actually topologically invariant: if $\phi \to \delta\phi$ and $\hat{\varphi} \to \delta\hat{\varphi}$ then $g \to g$ is invariant

(ii) g is additive, so it must be proportional to the winding number

The 't Hooft - Polyakov monopole

Assume that we are at winding number 1. This means
$$\text{as } |x| \to \infty \quad \phi_i = \langle\phi\rangle \hat{x}_i F(r)$$

Ansatz $A^i_j = \frac{\varepsilon_{jik}\hat{x}_k}{e} \frac{G(r)}{r}$.

We have $F(r) \to 1$ as $r \to \infty$.

Then $\frac{1}{2}(D_i\vec{\phi})^2 = \langle\phi\rangle^2 \left[\frac{F^2(1-G)^2}{r^2} + \frac{F'^2}{2} \right]$

which means we need $G(r) \to 1$ and $F'(r) \to 0$

Also $\frac{1}{4}(F^a_{ij})^2 = \frac{1}{e^2}\left[\frac{G'^2}{r^2} + \frac{(2G-G^2)^2}{2r^4} \right]$

so we also need $G'(r) \to 0$

This all means that $F^a_{ij} \to -\frac{\varepsilon_{ijk}\hat{x}_k\hat{x}_a}{er^2}$

and $B_i = \frac{1}{2}\varepsilon_{ijk}F'_{jk} \to \frac{1}{2}\varepsilon_{ijk}\hat{\phi}_a F^a_{jk} \to -\frac{\hat{x}_i}{er^2}$

hence $\boxed{g = -\frac{1}{e}}$

If the monopole solution was of winding number ν, $g = -\frac{\nu}{e}$

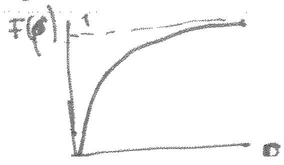
<u>The BPS monopole</u>

For $\lambda \ll e^2$, i.e. when $V(\phi)$ is small, one can derive a Bogomolnyi bound $E = \int d^3x\, \mathcal{H} \geq 4\pi\langle\phi\rangle|g|$

and saturate it by imposing $F^a_{ij} = \pm\varepsilon_{ijk}D_k\phi_a$

(which is a first order differential equation). The BPS solution is then $F = \coth(\rho) - \frac{1}{\rho}$ $\quad G = 1 - \frac{\rho}{\sinh(\rho)}$ $\quad \rho = e\langle\phi\rangle\cdot r$

Note that $\phi_i = \langle\phi\rangle\hat{x}_i F(r)$, so when $r \to 0$ $F(r) \to 0$



This means that at $r \to 0$ the symmetry $SU(2)$ is restored (the field solution are zero there!).

Monopoles in real life

- The Georgi-Glashow model was excluded as a model for e/w symmetry breaking because it does not accomodate neutral curren (that are observed): in other words there is no Z boson.
- In $SU(2) \times U(1)$ there are no monopoles (because the gauge group is not simply connected)
- In $SU(5)$ GUT theories, monopoles can exist, before the phase transition that breaks $SU(5)$ to $SU(3) \times SU(2) \times U(1)$

But then they should have been there in the early universe. If so, any early universe model has to explain how come we don't see them today in great abundance. So even if they don't exist, monopoles constrain our theories!