LECTURE 8

1. Review and some extras on anomalies

We have seen that, in theories with a Lagrangian

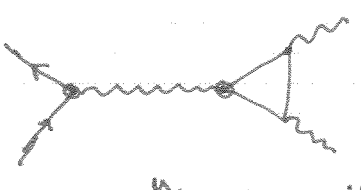
$$\bar\psi\, i \not{D}\, \psi \qquad\qquad D_\mu = \partial_\mu + ie A_\mu$$

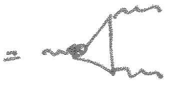
(where ψ is a Dirac fermion), the symmetry $\psi \to e^{i(\theta + \gamma_5 \theta_A)}\psi$ corresponds to a current $J_5^\mu = \bar\psi \gamma^\mu \gamma_5 \psi$ that is classicaly conserved. However, at the quantum level, the current is <u>not</u> conserved, due to the axial anomaly: when J_5^μ interacts with two photons the triangle diagrams violate the Ward identity, as can be seen by direct evaluation, where one has to be careful with extending γ_5 to d-dimensions.

If, in the original Lagrangian $\psi = \begin{pmatrix}\psi_u \\ \psi_d\end{pmatrix}$, an isospin doublet then the global symmetry transformation can be $\psi \to e^{i(\vec\theta\cdot\vec t + \gamma_5 \vec\theta_A \cdot \vec t)}\psi$ and the violated axial currents have an isospin index, $J_5^{\mu,i} = \bar\psi \gamma^\mu \gamma_5 t^i \psi$.

How can such currents appear in scattering processes?

a) If we insist on our theory being renormalizable, then the operator $\bar\psi \gamma^\mu \gamma_5 \psi$ is of mass dim. 3, and a Lorentz vector, so it should be coupled to some vector field V_μ, not necessarily a gauge field. Then the vector field would induce diagrams like



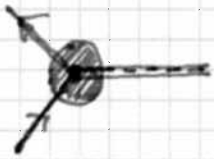
where the Ward identity for V_μ would not hold (i.e. $q_\mu \cdot M^\mu \neq 0$ where $M^\mu = \tilde{M}^{\mu\nu\lambda}\varepsilon^*_\nu \varepsilon^*_\lambda$, $\tilde{M}^{\mu\nu\lambda} =$ , even when $q^2 = 0$

(b) In a non-renormalizable, effective theory we could also have

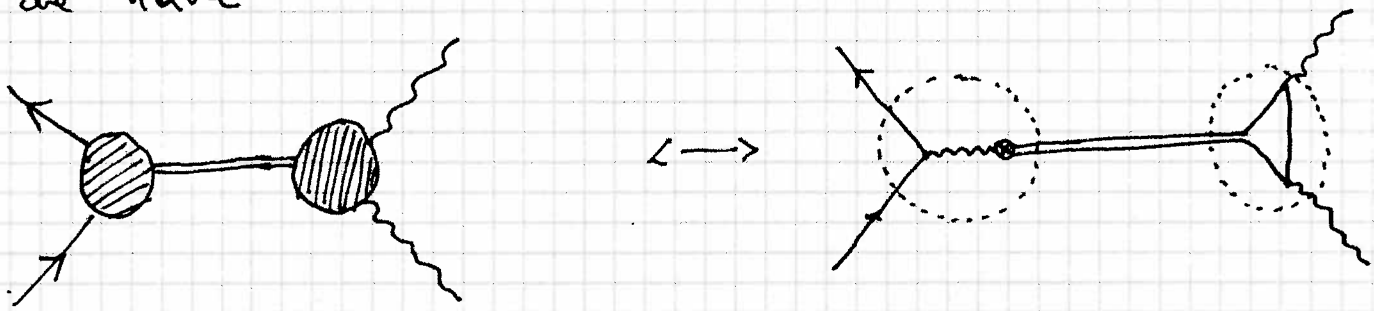
a $\bar{\psi}\gamma_\mu\gamma_5\psi(\partial^\mu\pi)\cdot f$ term, where π is a scalar field.

Such scalar fields appear as a consequence of spontaneous symmetry breaking of the (global) symmetry, and describe, in the effective Lagrangian, bound states of odd parity $\bar{\psi}_L\psi_R$.

This operator induces diagrams like



The scalar field couples to the e/m field tensor via a $g\pi F\tilde{F}$ operator, corresponding to the anomaly in terms of currents, so we have



Question: what is the source of $g\pi F\tilde{F}$ operator (and hence how strong is the coupling "g")?

1) The operator is allowed by the (broken) symmetries of the Lagrangian

2) It could be a consequense of S.S.B. only. In that case, because the π field corresponds to the breaking of the chiral, but not the U(1) symmetry of e/m, the coupling would be far too small compared to measurement

3) It actually is a consequense of the anomaly. This can also be seen by the effect that the chiral trs. has on the measure of the path integral: it induces a term $g\,\pi^0 F\tilde{F}$ (in the EFT language).

Note that the anomaly is proportional to $\frac{1}{2}\mathrm{Tr}\left[\{t^a, t^b\} t^c\right]$ where t^a, t^b, t^c are the generators of the symmetries that correspond to the currents that enter the triangle. So in the axial-isospin current with the photons we have $t^a = Q_f$, $t^b = Q_f$, $t^c = \frac{\sigma_3}{2} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and

$$\{t^a, t^b\} t^c = \frac{1}{2}\begin{pmatrix} Q_u^2 & 0 \\ 0 & -Q_d^2 \end{pmatrix}$$ whose trace is $\frac{1}{2}(Q_u^2 - Q_d^2) = \frac{1}{2}\left(\left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right) = \frac{1}{2}\,\frac{1}{3}$

(for every color and every fermion generator in case of quarks).

The axial anomaly with gluons would be $\frac{1}{2}\mathrm{Tr}\left(\{t^a, t^b\}\frac{\sigma_3}{2}\right) = \frac{1}{2}\mathrm{Tr}[\{t^a, t^b\}]\,\underbrace{\mathrm{Tr}\frac{\sigma_3}{2}}_{=0}$

So you only get an axial anomaly if up and down fermions belong to different representations of the gauge group.

(c) What if $\bar{\psi}\gamma^\mu\gamma_5\psi$ is coupled to a gauge field, Z_μ?

This can happen if (as in the Standard Model), the gauge field couples to left and right handed fermions differently (the theory is chiral). Then ~~the~~ $\mathcal{L}$ contains $\bar{\psi}_L \gamma^\mu Z_\mu g_L \psi + \bar{\psi}_R \gamma^\mu Z_\mu g_R \psi$ which can be rewritten as $\bar{\psi}\gamma^\mu g_V Z_\mu \psi + \bar{\psi}\gamma^\mu\gamma_5 g_A Z_\mu \psi$

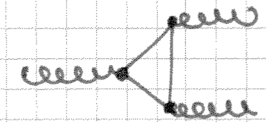
Now we have an axial current coupled to a gauge field. An anomaly would violate the Ward identity, hence there would be no gauge invariance any more!

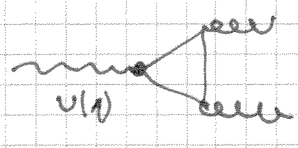
→ Any theory involving gauge bosons should be anomaly-free. ~~that only axial anomaly free~~. This means that for any combination of gauge currents in a triangle $\sum_{\text{species}} \mathrm{Tr}\left[\{t^a, t^b\} t^c\right]$ should be zero

~~→ If all matrices belong to~~

In the SM ~~the~~ gauge group content is $SU(3) \times SU(2)_L \times U(1)$

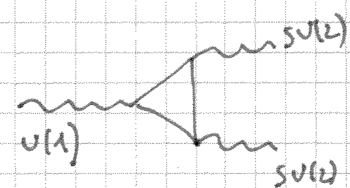
Possible anomalies

(i)  $SU(3) - SU(3) - SU(3)$: $\mathrm{tr}\left[\{t^a, t^b\}\, t^c\right] = 0$.

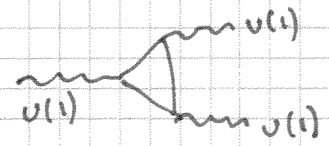
(ii) 

$SU(3) - SU(3) - U(1)$: $\sum\limits_{\text{doublets}} \mathrm{tr}\, \{t^a, t^b\}\, \cancel{Y}^c \sim \sum\limits_{\cancel{\text{doublets}}} Y$

$$= \underbrace{-\frac{1}{6}}_{u_L} \underbrace{-\frac{1}{6}}_{d_L} \underbrace{+\frac{2}{3}}_{u_R} \underbrace{-\frac{1}{3}}_{d_R}$$

(iii) 

$SU(2) - SU(2) - U(1)$ $\quad \underbrace{\sum 3}_{\text{color}} \left(\underbrace{-\frac{1}{6} \cdot \frac{1}{2}}_{u_L} \underbrace{- \frac{1}{6} \cdot \frac{1}{2}}_{d_L} \right) + \underbrace{\frac{1}{2}}_{e_L}$

(iv) 

$U(1) - U(1) - U(1)$ $\quad \sum 3 \cdot \left(\underbrace{\left(-\frac{1}{6}\right)^3}_{u_L} \underbrace{- \left(\frac{1}{6}\right)^3}_{d_L} \underbrace{+ \left(\frac{2}{3}\right)^3}_{u_R} \underbrace{- \left(\frac{1}{3}\right)^3}_{d_R} \right) + \underbrace{\left(\frac{1}{2}\right)^3}_{\nu_L} + \underbrace{\left(\frac{1}{2}\right)^3}_{e_L} + \underbrace{(-1}_{e_R}$

$$= 0$$

$\longrightarrow$ We see that cancelation of the $U(1) - U(1) - U(1)$ anomaly requires families of quarks and leptons

SM	SU(3)	SU(2)	U(1)
$\begin{pmatrix} u \\ d \end{pmatrix}_L$	3	2	$-1/6$
u_R^*	$\bar{3}$	1	$2/3$
d_R^*	$\bar{3}$	1	$-1/3$
$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	1	2	$1/2$
e_R^*	1	1	-1

Topological objects: solitons (aka kinks, hedgehogs)

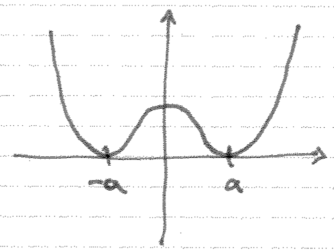
Let's consider a scalar φ^4 theory in $D=2$

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - U(\varphi)$$

$$= \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{\lambda}{2}\left(\varphi^2 - \frac{\mu^2}{\lambda}\right)^2$$

The potential $U(\varphi)$ has two degenerate minima at $\varphi = \pm a$

with $a = \frac{\mu}{\sqrt{\lambda}}$



The energy for a given φ is $\int dx\,(\partial_0\varphi)^2 + (\partial_1\varphi)^2 + U(\varphi)$

Question: are there time-independent solutions of the eq. of motion ~~that minimize the energy~~ $\cancel{E = \int dx\left[\frac{1}{2}(\partial_1\varphi)^2 + U(\varphi)\right]}$?

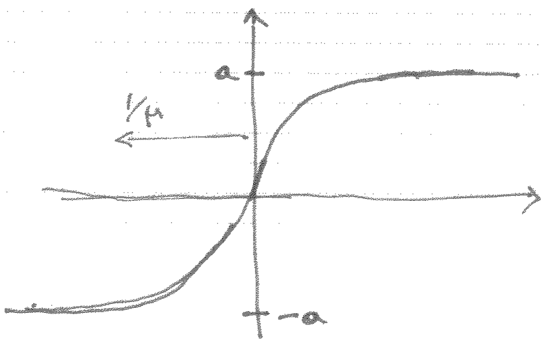
Trivial answer: $\varphi = \pm a$ everywhere, always. This has $E=0$

Non-trivial answer: $\partial_0\varphi = 0$, $\delta\left[\frac{1}{2}(\partial_1\varphi)^2 + U(\varphi)\right] = 0$

$$\Rightarrow \quad \frac{1}{2}(\partial_1\varphi)^2 - U(\varphi) = 0$$

$$\Rightarrow \quad \frac{d\varphi}{dx} = \sqrt{2U(\varphi)} \quad \Rightarrow \quad \frac{d\varphi}{\sqrt{2U(\varphi)}} = dx$$

$$\Rightarrow \quad \varphi = a\tanh(\mu x)$$



This solution interpolates between

$\varphi = -a$ at $x = -\infty$

and

$\varphi = +a$ at $x = +\infty$.

It is called a kink / soliton. Another solution, the anti-soliton, can be found by $\varphi_0 = -a\tanh(x\mu)$.

Can we find solutions that are not centered at zero? Sure, $\varphi = a\tanh(\mu($

The energy of the soliton is $E = \int dx \frac{1}{2}(\partial_\varphi \varphi)^2 + U(\varphi)$

$$= \int dx\, (\partial_1 \varphi)^2 = \int dx \frac{d\varphi}{dx}\frac{d\varphi}{dx}$$

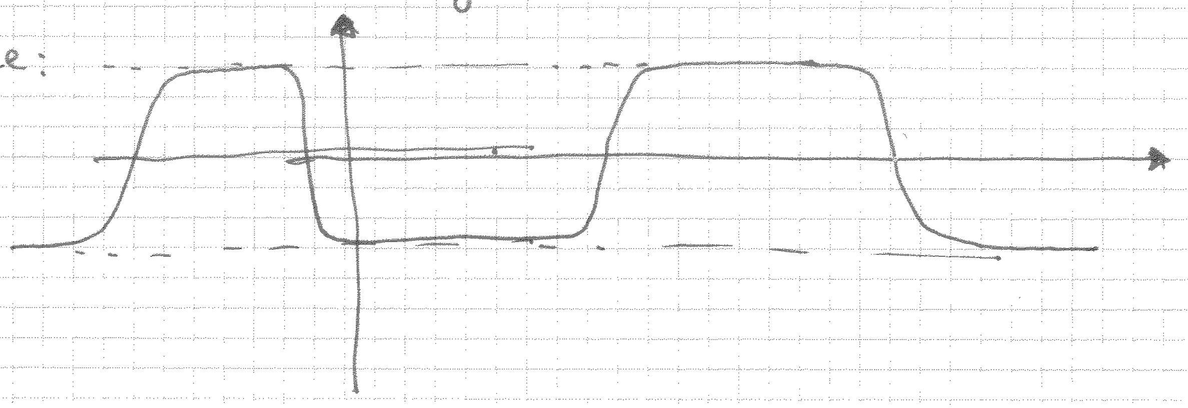
and since $\frac{d\varphi}{dx} = \sqrt{2U(\varphi)}$ we have

$$E = \int d\varphi \sqrt{2U(\varphi)} = \dots = \frac{4\mu^3}{3\lambda}$$

One can show (see Coleman, Aspects of symmetry p. 192) that any small perturbation around the solution $\varphi(x) = a\tanh(\mu x) + \delta(x,t)$ has energy larger than ϕ_0, hence the solution $\phi_0(x) = a\tanh(\mu x)$ is stable (it doesn't dissipate).

Can we find solutions with more than one soliton?

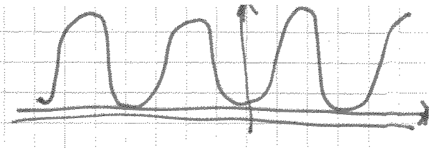
Since we want the energy to be minimal, the behavior at $x\to\pm\infty$ should be $\varphi(\pm\infty) \to \pm a$, i.e to one of the trivial vaccua. As long as the solitons are far away from each other we can have as many as we like:



The equations of motion for the field are

$$\partial^2\varphi + \frac{dU}{d\varphi} = 0 \Rightarrow \partial^2\varphi + \lambda(\varphi^2 - a^2)\cdot 2\varphi = 0$$

An alternative potential is $U(\varphi) = \frac{a}{b^2}(1-\cos b\varphi)$



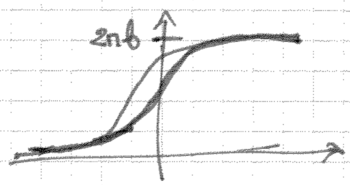
for which $\frac{dU}{d\varphi} = \frac{a}{b} \sin(b\varphi)$

and the corresponding equation is

$$\partial^2\varphi + \frac{a}{b}\sin b\varphi = 0$$ called the Sine-Gordon eq.

The soliton solution is $\varphi_0 = \frac{4}{b}\arctan\left(e^{x\sqrt{a}}\right)$ and

has energy $E = \frac{8\sqrt{a}}{b^2}$



Derrik's theorem

Are there similar, time-independent solutions in more dimensions?

The answer is: NO! Demanding that $\frac{1}{2}(\nabla\varphi)^2 + U(\varphi)$ is stationary

means that if I define $\varphi(x) \to \phi(\lambda x)$ then

$$\frac{d\varphi}{dx} = \frac{d\varphi}{d(\lambda x)} \cdot \lambda$$

so $\int d^dx \; \frac{1}{2}(\nabla\varphi)^2 + U(\varphi) \to \int \frac{d^d(x\lambda)}{\lambda^d} \; \lambda^2 \frac{1}{2}(\nabla\varphi)^2 + U(\varphi)$

and $\frac{\delta}{\delta\lambda} \int d^dx \left(\frac{1}{2}(\nabla\varphi)^2 + U(\varphi)\right)\Bigg|_{\lambda=1} = (2-D)\int d^dx \; \frac{1}{2}(\nabla\varphi)^2 - D\int d^dx \; U(\varphi)$

$= 0$

So if $D=1$, $\int d^dx \; \frac{1}{2}(\nabla\varphi)^2 - U(\varphi) = 0$

$$\Rightarrow \frac{1}{2}(\nabla\varphi)^2 = U(\varphi)$$

but if $D \geq 2$ then $\int d^dx \; \frac{1}{2}(\nabla\varphi)^2 (D-2) + D\,U(\varphi) = 0$

$\Rightarrow \int d^dx \, \nabla\varphi = 0$ and $\int d^dx \, U(\varphi) = 0$ $\Rightarrow$ only trivial solutions exist.

Note that this means nothing for time-dependent solutions!

These solutions to the eqs. of motion have many interesting characteristics

(i) They refer to non-linear eqs. of motion

(ii) The solutions interpolate between degenerate vaccua

(iii) They are of non-trivial topology (in this case the boundary of the one-d. space is discrete and "topology" has little meaning, but the existense of the solutions crucially depends on the boundary at infinity being non-zero

For example, $$E[\varphi] = \int_{-\infty}^{+\infty} dx \quad \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 + U(\varphi)$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} dx \left(\frac{d\varphi}{dx} - \sqrt{U(\varphi)\cdot 2}\right)^2 + \frac{1}{2}\int dx \frac{d\varphi}{dx}\sqrt{2U(\varphi)}$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} dx \underbrace{\left(\frac{d\varphi}{dx} - \sqrt{U(\varphi)\cdot 2}\right)^2} + \underbrace{\frac{1}{2}\int_{\varphi(-\infty)}^{\varphi(+\infty)} df \sqrt{2U(f)}}_{\text{topological charge}}$$

$$\Rightarrow \quad E[\varphi] \geq \frac{1}{2}\int_{\varphi(-\infty)}^{\varphi(+\infty)} df \sqrt{2U(f)}$$

with the equality saturated if $\frac{d\varphi}{dx} = \sqrt{2U(\varphi)}$.