Instantons

Consider a gauge theory with ~~Euclidean~~ Euclidean action

$$S[A] = \int d^d x \ \frac{1}{4} F^a_{ij} F^a_{ij}$$

We will look for ~~...~~ configurations that are (a) non-trivial topologically and (b) minimi[ze] the action. Expanding around such configuratio[ns] will be a quadratic field action, with first order terms vanishing and second order terms with righ[t] sig[n].

If $S[A]$ is finite, F^α_{ij} must vanish as $|x| \to \infty$. It is possible to find configurations where $A^\alpha_{ij}(x)$ vanishes as slowly as $\frac{1}{|x|}$, as long as the field goes to a pure gauge when $|x| \to \infty$

$$\lim_{|x|\to\infty} A^a_i t^\alpha = g^{-1}(\hat{x})\, \partial_i g(\hat{x})$$

note: we should have said $A^a_i t^a \to \frac{1}{g_s} g^+ \partial_i g$
we have absorbed $1/g$ ...

where $g(\hat{x})$ is an element of the gauge group G which does not depend on the magnitude of x (but only on the direction of x).

→ Since topologically non-trivial field configurations ~~can be continuously~~ have a limiti[ng] behaviour at ∞ which only depends on the direction, ~~one can~~ for every field $A(x)$ in a topological class we can define $A'(x) = A(x/R)/R$. Then

$$S[A'] = \frac{1}{4} \int d^d x \ \underline{F^\alpha_{ij}(x/R)\, F^\alpha_{ij}(x/R)}_{R^4} \overset{x=yR}{=} \frac{R^{d-4}}{4} \int d^d y \ F^\alpha_{ij}(y) F^\alpha_{ij}(y)$$

If $S[A]$ was a minimum, then, for $D \neq 4$ we can find another minimum, smaller than the previous, which is a contradiction ~~We can continuously take $R \to 0$, so if $d \neq 4$, $S[A'] \to 0$~~, which means $S[A] = 0$ which means $A^\alpha_{ij} = 0$ everywhere, which is the "trivial" configuration. So we can only hope to find a non-trivial topological solution for $d = 4$.

→ We want to analyze the topology of ~~field~~ configurations like $g(\hat{x})$ which are mappings from the sphere at infinity, S_3, parametrized by 3 angles, to the (representation of the) Lie group $SU(N)$. So $g(\hat{x})$ is a matrix in $SU(N)$ that depends on 3 angles (if $D=4$). For a general map from $S_d \to M$ there is a quantity, called the Cartan-Maurer form,

$$I[g] = \int d\theta_1 d\theta_2 .. d\theta_d \ \varepsilon^{i_1 .. i_d} \, \mathrm{Tr}\left[g^{-1} \frac{\partial g}{\partial \theta^{i_1}} \, g^{-1} \frac{\partial g}{\partial \theta^{i_2}} \cdots g^{-1} \frac{\partial g}{\partial \theta^{i_d}} \right]$$

with the amazing properties that

(1) It is invariant under coordinate trs. $\{\theta\} \to \{\theta'\}$

(2) It is invariant under ~~field~~ deformations of the mapping $g \to g + \delta g$

This immediately means that $I[g]$ is topological invariant: it has the same value for any field in the same topological class (homotopy class)

In our case we have $D=4$, so the maps are from $S_3 \rightarrow S_3$ (because we will consider that $g(\hat{x})$ is in an $SU(2)$ subgroup of $SU(N)$)

Then $$I[g] = \int d\theta_1 d\theta_2 d\theta_3 \; \varepsilon^{ijk} \, Tr\left[g^{-1} \frac{\partial g}{\partial \theta_i} \, g^{-1} \frac{\partial g}{\partial \theta_j} \, g^{-1} \frac{\partial g}{\partial \theta_k} \right]$$

$$= \int d\theta_1 d\theta_2 d\theta_3 \; \varepsilon^{ijk} \frac{\partial x^a}{\partial \theta_i} \frac{\partial x^b}{\partial \theta_j} \frac{\partial x^c}{\partial \theta_k} \underbrace{Tr\left[g^{-1} \frac{\partial g}{\partial x^a} \, g^{-1} \frac{\partial g}{\partial x^b} \, g^{-1} \frac{\partial g}{\partial x^c} \right]}_{\lim_{|x| \to \infty} Tr\left[A_a A_b A_c\right]}$$

Defining $$G_m = \varepsilon_{mijk} \left[A^a_i F^a_{jk} - \frac{1}{3} f^{abc} A^a_i A^b_j A^c_k \right]$$

$$= \frac{2}{N} \varepsilon_{mijk} \, Tr\left[A_i F_{jk} + \frac{2i}{3} A_i A_j A_k \right]$$

$Tr\, t^a t^b = \frac{N}{2} \delta_{ab}$

$[t^a, t^b] = i f^{abc} t^c$

Then $$\oint dS \; G_m \cdot \hat{x}_m = I[g] \frac{4}{3N}$$

$$\Rightarrow I[g] = \int d^4x \; \partial_m \cdot G_m \; \frac{3N}{4}$$

But $$\partial_m G_m = \frac{1}{4} F^a_{ij} \tilde{F}^a_{ij} = \frac{2}{N} Tr\, F \tilde{F}$$ ~~(for N=2)~~.

with $$\tilde{F}^a_{ij} = \frac{1}{2} \varepsilon_{ijkm} F^a_{km}$$

So $$I[g] = \int d^4x \, \frac{2}{N} Tr(F \tilde{F}) \cdot \frac{3N}{4}$$

The Bogomol'nyi inequality

$$\int d^4x \left(F^a_{ij} \mp \tilde{F}^a_{ij}\right)^2 \geq 0$$

$$\Rightarrow \quad \int d^4x \quad F^a_{ij} F^a_{ij} \mp 2\, F^a_{ij} \tilde{F}^a_{ij} + \tilde{F}^a_{ij} \tilde{F}^a_{ij} \geq 0$$

$$\Rightarrow \quad \int d^4x\, F^a_{ij} F^a_{ij} \geq \pm \int d^4x\, F^a_{ij} \tilde{F}^a_{ij}$$

$$\Rightarrow \quad \frac{1}{4} \int d^4x\, F^a_{ij} F^a_{ij} \geq \pm \frac{\ }{4} \int d^4x\, F^a_{ij} \tilde{F}^a_{ij} = \pm \frac{2}{4} \frac{\ }{N} \int d^4x\, \mathrm{Tr}$$

$$\boxed{S[A] \geq \pm \frac{1}{3N} I[g].}$$

So the minimum for $S[A]$ is reached when $S[A] = \dfrac{|I[g]|}{3N}$

which happens when $F^a_{ij} = \pm \tilde{F}^a_{ij}$.

So we are looking for a configuration that

1) has a limit at ∞ $\quad A_i \to g^{-1} \partial_i g$

2) has a non-vanishing $I[g]$

3) is self dual or anti-self dual.

The Belavin - Polyakov - Schwarz - Tyupkin (BPST) solution

$$A_i(x) = -i \frac{x^2}{x^2 + R^2} \quad g_*^{-1} \partial_i g_*$$

with $\quad g_* = \dfrac{x_4 + i\, x_i \sigma_i}{|x|} \qquad i = 1..3.$

Obviously as $|x| \to \infty$ $\quad A_i \to g_*^{-1} \partial_i g_*$. It can be shown that

$I[g_*] = 24\pi^2 N$, which means (a) it's in the winding number 1 class and (b) $S[A] = 8\pi^2$

note the BPST instanton is selfdual. To get the anti-selfdual you need to send $x_i \to -x_i$

Note that $g_*^{-1} = \dfrac{x_4 - i x_i \sigma_i}{|x|} = g_*^\dagger$

and that
$$A_i = \begin{cases} \left(\dfrac{x_4 - i x_j \sigma_j}{x^2+R^2}\right)\sigma_i & i=1,2,3 \\ -\dfrac{x_j \sigma_j}{x^2+R^2} & i=4. \end{cases}$$

so A_i falls as $\dfrac{1}{|x|}$ at infinity, as promised. It is a configuration localized at $x=0$, of size R.

Solutions for other winding numbers can be found by superimposing BPST instantons centered far away from each other. For a general winding number $S[A] = 8\pi^2 |\nu|$

→ These results are valid for a normalization where we have absorbed a factor of $1/g^2$, so in reality, the Euclidean action is $S[A] = 8\pi^2/g^2$ and the ~~Minkowski action is~~ $\cancel{-\dfrac{8\pi^2|\nu|}{g^2}}$ contribution to the Euclidean path integral is

$$e^{-8\pi^2|\nu|/g^2}$$

This means (a) the instanton contribution is not perturbative: for $g=0$ (and all derivatives in an expansion around $g=0$) the contribution vanishes

(b) as g is a running coupling constant, for example, in QCD $g^2(\mu) = \dfrac{8\pi^2}{b_0 \log(\mu/\Lambda)}$ the relevant $\mu \approx 1/R$, and then $\exp(-8\pi^2|\nu|/g^2) = (R\Lambda)^{b_0}$

As $R \to 0$, $\frac{1}{R} \to \infty$, $\mu \to \infty$, $g \to 0$ (if $b_0 > 0$), in QCD $b_0 = 11 - \frac{10}{3}$ and the instanton contribution goes as $(R\Lambda)^{b_0}$. If $R \to \infty$, $\mu \to 0$ and $g \to \infty$, at which case the instanton contribution blows up.

Open questions: how do we expand around instanton solutions beyond LO? How do we count non-equivalent configurations in the path integral?

Appendix: Euclidean space

$$x^0 = -i\, x_4$$

$$A^0 = -i\, A_4$$

$$e^{iS} \to e^{-S_E}$$

t' Hooft symbols

$$\eta_{a\mu\nu} = \begin{cases} \varepsilon_{a\mu\nu} & a,\mu,\nu \in 1,2,3 \\ -\delta_{a\nu} & \mu = 4 \\ \delta_{a\mu} & \nu = 4 \\ 0 & \mu = \nu = 4 \end{cases}$$

Then the BPST instanton is $A_\mu^a = 2\, \eta_{a\mu\nu} \dfrac{x_\nu}{x^2 + \rho^2}$

or, in general

$$A_\mu^a = 2\, \eta_{a\mu\nu} \frac{(x - x_0)_\nu}{(x - x_0)^2 + \rho^2}$$

* The construction of the $\dfrac{x^2}{x^2+\rho^2}$ prefactor of the BPST, starting from the ansatz for g_*, and demanding self-duality can be found in Shifman 21.1