Lecture 6 Vacua in Yang-Mills theories and the strong CP problem

Let us consider pure SU(3) Yang-Mills theory (pure QCD)

$$\mathcal{L} = -\frac{1}{4} \operatorname{Tr}\left[G^{a}_{\mu\nu} G^{a}_{\mu\nu} \right] \qquad \text{with} \quad G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g\,[A_\mu, A_\nu]$$

$$A_\mu = A^a_\mu t^a \qquad a = 1, \dots 8$$

Due to gauge invariance, various field configurations connected by a gauge transformation, lead to the same physics; they are physically equivalent.

A gauge trs. sends $A_\mu \to U(x) A_\mu U^\dagger(x) + i\, U(x) \left(\partial_\mu U^\dagger(x)\right)$

where $U(x)$ is an SU(3) matrix dependent on x.

We will (partially) fix the gauge by demanding $A_0(x) = 0$.

This still leaves gauge freedom (there are 3 d.o.f. left)

Then $A_i(x) \to U A_i U^\dagger + i\, U \partial_i U^\dagger$

$A_0(x) \to U A_0 U^\dagger + i\, U \partial_0 U^\dagger$

so to keep $A_0(x) = 0$ we need to constrain ourselves to $U(\vec{x})$, a function of $\vec{x}$ only.

Question: Are there any field configurations that ~~minimize the action~~ make the Hamiltonian of the system vanish, except $A_i(\vec{x}) = 0$ for all $\vec{x}$?

$H = \int d^3x \, \frac{1}{2} \operatorname{Tr}(F_{ij} F^{ij})$

~~To minimize the action~~ we need $F_{ij}(\vec{x}) = 0$ for which it's only necessary that $A_i(\vec{x})$ is a "pure gauge", i.e.

$$A_i(\vec{x}) = U(\vec{x}) \partial_i U^\dagger(\vec{x})$$

Moreover, we want a finite classical action, so we need $U(\vec{x}) \to 1$ as $\vec{x} \to \infty$

The 3-d space becomes, then, a compactified R^3 as far as $U(\vec{x})$ is conserved, so $U(x)$ is a map from $S^3 \to G$ where G is the manifold of $SU(3)$.

Let us, for the moment, constrain ourselves to $U(\vec{x}) \in SU(2)$ subgroup of $SU(3)$. Then $G = S^3$, and we are looking at maps from $S^3 \to S^3$.

Intermezzo: homotopy groups $\pi_d(M)$: groups whose elements are equivalent classes of mappings from $S^d \to M$ that are continuously deformable among themselves.

$S^1 \to S^1$

The group is $U(1)$ in $d=1$ spatial dim.

1) $U(\theta) = e^{i\theta}$, $\theta \in S_1 = [0, 2\pi)$

The compactified R^1 is also S^1

2) standard mappings

$$U_0(\theta) = 1$$

$$U_1(\theta) = e^{i\theta}$$

$$U_n(\theta) = (e^{i\theta})^n = e^{in\theta}$$

3) $$n = \frac{i}{2\pi}\int_0^{2\pi} d\theta \; U \frac{d}{d\theta} U^\dagger$$

is invariant under deformations

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4) Define $K_\mu = \frac{i}{2\pi}\,\varepsilon_{\mu\nu} A_\nu$

where $A_\nu = i\, U\, \partial_\nu U^\dagger$

Chern-Simons current

Then $Q = \int d^2x \; \partial_\mu K_\mu$

$S^3 \to S^3$

The group is $SU(2)$ in $d=3$ spatial

1) $U(\vec{x}) = a(\vec{x}) + i\,\vec{b}(x)\cdot\vec{\sigma}$

with $a + |\vec{b}|^2 = 1$

so $M = S^3$. Also R^3 compactified is S^3

~~2) $U_1(\vec{x}) = -e^{i\pi \frac{\vec{x}\cdot\vec{\sigma}}{(\vec{x}^2+\rho^2)^{1/2}}}$~~

~~$U_n(\vec{x}) = [U_1(\vec{x})]^n$~~

2) $$U_1(x) = \frac{\vec{x}^2 - d^2}{\vec{x}^2 + d^2}\,\mathbb{1} + 2id\,\frac{\vec{x}\cdot\vec{\sigma}}{\vec{x}^2 + d^2}$$

3) $$n = \frac{i g_s^3}{24\pi^2}\int d^3x \; \mathrm{tr}[A_i A_j A_k]\,\varepsilon^{ijk}$$

with $A_i = iU\,\partial_i U^\dagger$

is invariant under deformations

4) $$K_\mu = \varepsilon^{\mu\nu\rho\sigma}\left[A^a_\nu \partial_\rho A^a_\sigma + f^{abc} A^a_\nu A^b_\rho \right.$$

Then $Q = \int d^4x \; \partial_\mu K_\mu$

Consider now a field configuration that at $t=-\infty$ has a winding number n_1 and at $t=+\infty$ has winding number n_2

Then the "instanton" has a topological charge Q

$$Q \equiv \frac{g_s^2}{32\pi^2}\int d^4x \; F^a_{\mu\nu} \; {}^*F^a_{\mu\nu} = \frac{g^2}{32\pi^2}\int d^4x \; \partial_\mu K^\mu$$

$$= \frac{g^2}{32\pi^2}\int dx^3 \; K^0 \Big|_{-\infty}^{+\infty} = n_2 - n_1 \neq 0 .$$

<u>Conclusion</u>: the vaccua of $SU(N)$ can be classified in homotopy classe

You cannot perform a continuous transformation from one class to the other ~~while~~ without passing from configurations that are not vaccua!

This means that the true vacuum is a superposition of the pre-vaccua with winding numbers $n_i \in \mathbb{Z}$. How to deal with this in the path integral?

<u>Heuristic derivation on Eucledian path integral</u>

The expectation value of a localized operator $\hat{O}(\phi)$ is

$$\langle \hat{O} \rangle_{\Omega} = \frac{\sum_n f(n) \int_n [d\phi] \, e^{I_\Omega[\phi]} \, O(\phi)}{\sum_n f(n) \int_n [d\varphi] \, e^{I_\Omega[\varphi]}}$$

Let's imagine $\Omega = \Omega_1 + \Omega_2$ and $O(\phi)$ is only non-zero at Ω_1 (large volume)

Then
$$\langle \hat{O} \rangle_{\Omega} = \frac{\sum_{n_1, n_2} f(n_1+n_2) \int_{n_1} [d\phi] \, e^{I_{\Omega_1}[\varphi]} \, O(\phi) \int_{n_2} [d\varphi] \, e^{I_{\Omega_2}[\varphi]}}{\sum_{n_1, n_2} f(n_1+n_2) \int_{n_1} [d\varphi] \, e^{I_{\Omega_1}[\varphi]} \int_{n_2} [d\varphi] \, e^{I_{\Omega_2}[\varphi]}}$$

For this to work we need $f(n_1+n_2) = f(n_1) \cdot f(n_2)$, i.e. $f(n) = e^{in\theta}$

Since $n = \frac{1}{64\pi^2} \int d^4x \; Tr(F\, {}^*F)$ we have an extra term in the Lagrangian: $\frac{\theta g^2}{32\pi^2} Tr(F \tilde{F})$

Note that θ is a constant in QCD (it defines the theory), not a variable.

Note also that $\theta \neq 0$ induces P and T violation. It's actually the only operator that can do so.

However P and CP = T are experimentally seen to be conserved to a very high degree: the neutron electric dipole moment is the stringest constraint. This means that $\theta \leq 10^{-9}$.

Why is θ so small? This is the "strong CP problem" in QCD. It spoils the naturalness of QCD.

Note: The chiral connection

We have seen that, once adding massless fermions, the QCD Lagrangian is chirally invariant. In particular, though, the axial current, corresponding to a transformation $\psi_f \to e^{i\alpha_f \gamma_5} \psi_f$ is inducing, via the path integral measure a new term in $\mathcal{L}$

$$[d\psi][d\bar\psi] \to [d\psi][d\bar\psi] \; e^{i/32\pi^2 \int d^4x \; Tr(F\tilde{F}) \sum_f \alpha_f}$$

This is entirely equivalent to $\theta \to \theta + 2 \sum_f \alpha_f$

Such a trans. would affect non-chirally invariant mass terms, so

$$\mathcal{L}_m = \sum_f \left(M_f \bar\psi_L \psi_R + M_f^* \bar\psi_R \psi_L \right)$$

would pick up $M_f \to e^{2i\alpha_f} M_f$.

Rotating away the θ results in complex phases for the mass matrices.