5 Chiral perturbation theory

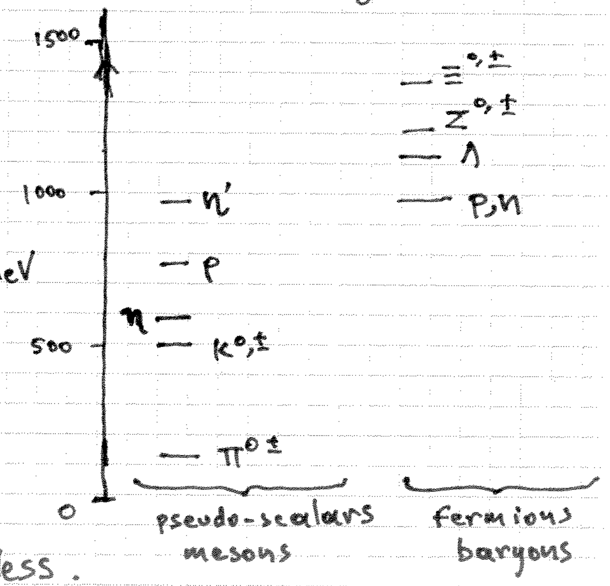
We currently believe that the fundamental constituents of all strongly interacting particles are quarks and gluons, described by a very simple Lagrangian

$$\mathcal{L} = i\,\bar{\psi}\,\not{D}\,\psi$$

$$D_\mu = \partial_\mu + i\,g_s\,\frac{\lambda_a}{2}\,G^a_\mu$$

However, because the strong interactions are strong, at low energies we only see bound states of quarks : mesons and baryons.

We know today that mesons are bound states of $q\bar{q}$, and the baryons color neutral combinations of three quarks.



We would like to have a field theoretical description of the mesons (and baryons) and their interactions, at relatively low energies of the order of the GeV or less.

* Note that bound states of quarks are qualitatively different than bound states at the atomic level, driven by the electromagnetic force: while an atom (of hydrogen) has a binding energy relatively low (~13.6eV) compared with the masses of its constituents (the proton, 938 MeV, and the electron, 0.511 MeV), in a QCD bound state, the binding energy is much larger than the sum of the masses of the constituents.

→ We have seen how (almost) massless pions can appear as (pseudo-) Goldstone bosons corresponding to the spontaneous breaking of the (approximate) chiral $SU(2)\times SU(2)$ symmetry.

To proceed, we would like to

1) enlarge the chiral symmetry group to accomodate the eight meson states of low ($< 1\,\text{GeV}$) energy.

2) add the baryon states in another octet (we only described (p) and (n), corresponding to $\psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix} \to \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$ up to now)

3) introduce a low-energy equivalent of the electroweak interaction to describe weak decays of mesons

We saw in the previous lecture how the language of non-linear representation of the Goldstone bosons in the σ-model allows us to write a Lagrangian that can be systematically expanded in terms of momentum exchanged, and to consistently decouple (i.e. remove) from the Lagrangian high mass fields without spoiling the low energy physics.

We now start by considering the QCD Lagrangian with 3 flavors.

$$\mathcal{L} = \bar{\psi}_L \, i \not{D} \, \psi_L + \bar{\psi}_R \, i \not{D} \, \psi_R$$

where $\psi_L = \begin{pmatrix} \psi_u \\ \psi_d \\ \psi_s \end{pmatrix}$ and $\not{D} = \begin{pmatrix} \not{D} & 0 & 0 \\ 0 & \not{D} & 0 \\ 0 & 0 & \not{D} \end{pmatrix}$

which is invariant under an $SU(3)_L \otimes SU(3)_R$ transformation

$$\psi_L \to e^{i \vec{\theta}_L \cdot \vec{\lambda}/2} \psi_L$$

$$\psi_R \to e^{i \vec{\theta}_R \cdot \vec{\lambda}/2} \psi_R$$

with λ_i the Gell-Mann matrices (the 8 generators of $SU(3)$ in the fundamental representation)

We immediately know how to describe the breaking of $SU(3)_L \times SU(3)_R \to SU(3)_V$ at low energies, where there are only Goldstone bosons:

Introduce $U = e^{i\sqrt{2}\,\phi/f}$ where $\phi = \phi_i \frac{\lambda_i}{2}$

Specifically $\quad \phi = \phi_i \dfrac{\lambda_i}{2} = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}$

As in the σ-model, we need $\quad U(\phi) \to U_R\, U(\phi)\, U_L^\dagger$

under $SU(3)_L \times SU(3)_R$. The ϕ_i transform highly non-linearly under the full group. However, they transform linearly under the surviving $SU(3)_V$ group.

Now, the low energy Lagrangian will be constructed by all possible terms that include derivatives and respect the full symmetry group. There will be an infinite series of terms in $\mathcal{L}$ involving higher and higher number of derivatives of U.

So
$$\mathcal{L} = \mathcal{L}_{(2)} + \mathcal{L}_{(4)} + \mathcal{L}_{(6)} + \dots$$

the lowest order term will be
$$\mathcal{L}_{(2)} = \frac{f^2}{4}\,\mathrm{Tr}\left[\partial_\mu U^\dagger \partial^\mu U\right]$$
$$= \dots = \frac{1}{2}\,\mathrm{Tr}\left[\partial_\mu \phi\, \partial^\mu \phi\right] + \frac{1}{12 f^2}\,\mathrm{Tr}\Big[\phi \overleftrightarrow{\partial_\mu} \phi\ \ \phi \overleftrightarrow{\partial^\mu}$$
$$+ \; O\left(\phi^6/f^4\right)$$

* Note that $U = e^{i \frac{\phi_i \lambda_i}{f\sqrt{2}}}$ is a 3×3 matrix, so taking derivati requires expanding $U(\phi)$ in series: handle with care!

* Note that f^2 was introduced to make the kinetic term come out right, but it also determines the scattering among mesons

* $\mathcal{L}_{(4)} = \alpha_1 \left[\mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U)\right]^2 + \alpha_2\, \mathrm{Tr}(\partial_\mu U^\dagger \partial_\nu U)\ \mathrm{Tr}(\partial^\mu U^\dagger \partial^\nu U)$

* In general, every term in the Lagrangian will involve the introduction of undetermined constants, that have to be input from experiment.

We now can trivially calculate various amplitudes for scatterin of mesons, like $\pi^0\pi^+ \to \pi^0\pi^+$, or even $\pi^0\pi \to \pi\pi\pi\pi$, $\pi\pi \to 6\pi$ etc. As in the σ-model case, $M_{\pi^0\pi^+ \to \pi^0\pi^+} = \frac{(P_+ - P_0)^2}{f^2}$

Next, we'd like to introduce mass terms in the QCD Lagrangian,

$$m\bar{\psi}\psi = m\bar{\psi}_L\psi_R + m\bar{\psi}_R\psi_L.$$

but we'll assume m is a 3x3 matrix, so the term is

$$\bar{\psi}_L m^+ \psi_R + \bar{\psi}_R m^\bullet \psi_L$$

This term would also be invariant under $SU(3)_L \otimes SU(3)_R$ if

$$m \to U_R m U_L^+$$

The correct way to introduce the masses to the effective Lagrangian, now describing pseudo-Goldstone bosons, is exactly pretending that the object m is actually an "external field χ that has the correct transformation properties to keep the QCD Lagrangian invariant under $SU(3)_L \times SU(3)_R$, but is frozen to its expectation value m. In other words, we need to add to $\mathcal{L}_{(2)}$ terms that ~~are~~ involve U and χ

$$\mathcal{L}_{(2)} = \frac{f^2}{4} \mathrm{Tr}\left[\partial_\mu U^+ \partial^\mu U + U^+\chi + \chi^+ U\right]$$

and then fixing $\chi = 2B_0 \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$

It is then easy to expand the last terms in ϕ^i and find the masses of the mesons as a function of the masses of up, down and strange quarks, and the constant B_0. This results at the Gell-Mann-Okubo mass formula

$$4 m_K^2 - 3 m_\eta^2 - m_\pi^2 = 0$$

and the Weinberg ratio of quark masses

$$\frac{2 m_K^2 - m_\pi^2}{m_\pi^2} = \frac{2 m_s}{m_d + m_u} \approx \frac{m_s}{m} \approx 25$$

where in the last step we assume $m_u \approx m_d$. The current Particle Data Group entry for $m_u = 2.3^{+0.7}_{-0.5}$ MeV, $m_d = 4.8^{+0.7}_{-0.3}$ M and $m_s = 95 \pm 5$ MeV, so $\frac{m_s}{\frac{m_u + m_d}{2}} = \frac{m_s}{m} = 27 \pm 1$

* The higher term in the expansion of $U^\dagger \chi + \chi^\dagger U$ ~~would~~ contribute a term $\frac{1}{6f^2} \mathrm{Tr}(\chi \phi^4)$ which is a correction to pion scattering

$$M_{\pi^+\pi^0 \to \pi^+\pi^0} = \frac{(P_+ - P_0)^2 - m_\pi^2}{f_0^2}$$

We would, now, like to couple our mesons to gauge fields. There can be two types of couplings, a vector type and an axial type: In general, the quark Lagrangian would be including terms like $\bar{\Psi}_L \gamma^\mu l_\mu(x) \Psi_L + \bar{\Psi}_R \gamma^\mu r_\mu(x) \Psi_R$. The natural way to do this is to promote the global $SU(3)_L \times SU(3)_R$ to local (gauge) symmetries:

$$\psi_L \to e^{i v_i \lambda_i} \psi_L \qquad \psi_R \to e^{i a_i \lambda_i} \psi_R$$

with $v_i(x)$, $a_i(x)$ now functions of x^μ. In order to keep the $i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + i\bar{\psi}_R \gamma^\mu \partial_\mu \psi_R$ term invariant one needs to introduce $D_{L\mu} = \partial_\mu - i\, l_\mu$ $\quad D_{R\mu} = \partial_\mu - i\, r_\mu$ where $l_\mu(x) = \partial_\mu v_i(x) \lambda_i$ and $r_\mu(x) = \partial_\mu a_i(x) \lambda_i$. Then the Lagrangian becomes $i\bar{\psi}_L \not{D}_L \psi_L + i\bar{\psi}_R \not{D}_R \psi_R$, ~~explicit~~ manifestly invariant under <u>local</u> $SU(3)_L \times SU(3)_R$.

In the non-linear parametrization, $U(\phi)$ transforms as ~~an object~~ ~~of~~ $U(\phi) \to U_R\, U(\phi)\, U_L^\dagger$, so the covariant derivative here will act as $D_\mu U \to D_\mu'\, U_R U U_L^\dagger$ and to preserve the previous frs. property we need $D_\mu' = U_L D_\mu U_R^\dagger$, which is achieved if $D_\mu U = \partial_\mu U - i\, l_\mu U + i\, U r_\mu$ (note that l_μ, r_μ are 3×3 matrices, like U).

Then, neglecting kinetic terms of the l_μ, r_μ fields, we can write down the new Lagrangian

$$\mathcal{L}_{(2)} = \frac{f^2}{4} \mathrm{Tr}\left[D_\mu U^\dagger D^\mu U + U^\dagger \chi + \chi^\dagger U \right]$$

Expanding this one can find the explicit interaction terms between mesons and the fields l_μ, r_μ. In the case of the electroweak standard model sector,

$$r_\mu = -e A_\mu Q \qquad \text{with} \quad Q = \frac{1}{3}\begin{bmatrix} 2 & & \\ & -1 & \\ & & -1 \end{bmatrix}$$

$$l_\mu = -e A_\mu Q - \frac{e}{\sqrt{2}\sin\theta_W} W_\mu^+ T^+ \quad \text{with} \quad T^+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In the case of the QCD sector $\quad r_\mu = l_\mu = g_s\, G^a_\mu\, \lambda^a$

Note that the electroweak bosons couple differently to left and right chiral fermions, $l_\mu \neq r_\mu$. In such a case we say we have a <u>chiral</u> theory. The strong interaction gauge bosons, the gluons, couple identically to left and right ~~quark~~ fermions, hence the pure QCD interactions are not chiral (although they preserve chiral symmetry, at least before symmetry breaking considerations — the terminology can, at times, be confusing).

Finally, defining $\quad u = U^{1/2}\, e^{i\,\xi\cdot\lambda/2v}\quad$, and

$$u_\mu = i\left[u^\dagger D_{L_\mu} u - u\, D_{R_\mu} u^\dagger\right]$$

we can construct a meson-baryon Lagrangian, starting with

$$\mathcal{L}^{(1)}_{MB} = \mathrm{Tr}\left[\bar{B}\,(i\not{D} - M)\,B + \frac{d}{2}\,\bar{B}\,\gamma^\mu\{u_\mu, B\} + \frac{f}{2}\,\bar{B}\,\gamma^\mu\gamma_5[u_\mu, B]\right]$$

with $\quad B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix}$