The σ-model

Introduction: what happens to our pion theory when more than one pion is involved in the interaction? Current algebra becomes complicated. We want to look closer to an effective description of the system. We start by studying a toy model for spontaneous symmetry breaking of chiral symmetry

(a) linear representation

$$\mathcal{L} = \bar\psi i \not\partial \psi + \frac{1}{2}\partial_\mu \tilde\sigma \partial^\mu \tilde\sigma + \frac{1}{2}\partial_\mu \vec\pi \partial^\mu \vec\pi - \frac{\lambda}{4}\left(\tilde\sigma^2 + \vec\pi^2 - v^2\right)^2 - g\bar\psi\left(\tilde\sigma + i\vec\sigma\cdot\vec\pi\gamma_5\right)$$

$$\psi = \begin{pmatrix} p \\ n \end{pmatrix} \text{ or } \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$$

Let's define $\Sigma = \tilde\sigma \mathbb{1} + i\vec\sigma\cdot\vec\pi$

Then $\frac{1}{2}\mathrm{tr}(\Sigma^\dagger \Sigma) = \tilde\sigma^2 + \vec\pi^2$, $\quad \tilde\sigma = \frac{1}{2}\mathrm{Tr}\,\Sigma$, $\quad \pi^k = -\frac{i}{2}\mathrm{Tr}(\sigma^k\Sigma$

$$\frac{1}{4}\mathrm{tr}\left(\partial_\mu\Sigma^\dagger \partial^\mu \Sigma\right) = \frac{1}{2}\partial_\mu\tilde\sigma\partial^\mu\tilde\sigma + \frac{1}{2}\partial_\mu\vec\pi\partial^\mu\vec\pi$$

The Lagrangian can then be written as

$$\mathcal{L} = \bar\psi_L i\not\partial\psi_L + \bar\psi_R i \not\partial \psi_R + \frac{1}{2}\mathrm{tr}\left(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\right) - \frac{\lambda}{4}\left(\frac{1}{2}\mathrm{tr}(\Sigma^\dagger\Sigma) - v^2\right)^2$$
$$- g\bar\psi_R\Sigma^\dagger\psi_L - g\bar\psi_L\Sigma\psi_R$$

It is invariant under $SU(2)_L \times SU(2)_R$ if $\Sigma \to U_L \Sigma U_R^\dagger$

$\psi_L \to U_L\psi \qquad U_{L,R} = e^{-i\vec a_{L,R}\cdot\vec\sigma/2}$

$\psi_R \to U_R\psi$

Note: the fields $\tilde\sigma$ and π^i transform as
$$\tilde\sigma \to \tilde\sigma + \frac{1}{2}(\vec a_L - \vec a_R)\cdot\vec\pi$$
$$\vec\pi \to \vec\pi - \frac{1}{2}(\vec a_L - \vec a_R)\tilde\sigma - \frac{1}{2}\vec\pi\times\vec a$$

In particular, under isospin rotations, $\vec a_L = \vec a_R = \vec a$ $\;\tilde\sigma$ and $\vec\pi$ do not m

If $v^2 > 0$, the minimum of the potential is at $\tilde\sigma_*^2 + \vec\pi_*^2 = v^2$.

We chose $\tilde\sigma_* = v$, $\vec\pi_* = \vec 0$.

Define the shifted field $\hat{\sigma} = \tilde{\sigma} - v$. Then the vacuum is at $\hat{\sigma} = 0$. Introducing this to the Lagrangian we have

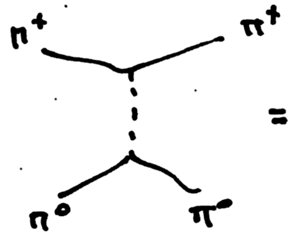
$$\mathcal{L} = \bar{\psi}\, i \not{\partial} \psi + \frac{1}{2} \partial_\mu \hat{\sigma}\, \partial^\mu \hat{\sigma} + \frac{1}{2} \partial_\mu \vec{\pi}\, \partial^\mu \vec{\pi} - \frac{\lambda}{4} \left[\hat{\sigma}^4 + (\vec{\pi}^2)^2 + 4 \hat{\sigma}^2 v^2 + 2 \hat{\sigma}^2 \vec{\pi}^2 + 4 \hat{\sigma}^3 v + 4 \vec{\pi}^2 \hat{\sigma} v \right] - g \bar{\psi} (\tilde{\sigma} + i\, \vec{\tau} \cdot \vec{\pi} \gamma_5) \psi - g \bar{\psi} \psi \cdot v$$

We have a mass term for the fermionic fields, $m_\psi = g \cdot v$, a mass term for the $\hat{\sigma}$ fields, $\frac{1}{2} 2\lambda v^2 \hat{\sigma}^2$, and interaction terms between the scalar fields $-\frac{\lambda}{4} \hat{\sigma}^4$, $-\frac{\lambda}{4} (\vec{\pi}^2)^2$, $-\frac{\lambda}{2} \hat{\sigma}^2 \vec{\pi}^2$, $-\lambda v \hat{\sigma}^3$, $-\lambda v\, \vec{\pi}^2 \cdot \hat{\sigma}$

The $SU(2) \times SU(2)$ symmetry is spontaneously broken by the mass terms down to $SU(2)_V$ (i.e. with $\vec{a}_L = \vec{a}_R$), and three massless G.B. survive, the "pions" $\vec{\pi}$.

$\pi^+ \pi^0$ scattering

$= -2i\lambda$



$$= (-2i\lambda v)^2 \frac{i}{p_\sigma^2 - \underbrace{2\lambda v^2}_{m_\sigma}}$$

$$\mathcal{M} = -2i\lambda + \frac{(-4\lambda^2 v^2 i)}{p_\sigma^2 - 2\lambda v^2} = -2i\lambda \left[1 + \frac{2\lambda v^2}{p_\sigma^2 - 2\lambda v^2} \right]$$

$$= -2i\lambda \left[1 - \frac{1}{1 - p_\sigma^2 / 2\lambda v^2} \right] = -2i\lambda \left[\frac{p_\sigma^2}{2\lambda v^2} + \left(\frac{p_\sigma^2}{2\lambda v^2} \right)^2 + \cdots \right]$$

→ As $p_\sigma^2 \to 0$ the pion interaction switches off and the only remnant of the pions is through the nucleon-nucleon interaction

→ For this to happen, a cancelation between  and  takes place

→ For low energy interactions we would like to decouple the massive $\hat{\sigma}$ field, but we cannot do it because without it the cancellation is incomplete.

(8) Non-linear representation

starting from $\mathcal{L} = \bar{\psi}_L i \not{\partial} \psi_L + \bar{\psi}_R i \not{\partial} \psi_R + \frac{1}{4} \operatorname{tr}(\partial_\mu \Sigma^+ \partial^\mu \Sigma)$

$$- \frac{\lambda}{4} \left(\frac{1}{2} \operatorname{tr}(\Sigma^+ \Sigma) - v^2 \right)^2 - g \bar{\psi}_R \Sigma^+ \psi_L - g \bar{\psi}_L \Sigma \psi_R$$

we wrote $\Sigma = \tilde{\sigma} + i \vec{\sigma} \cdot \vec{n}$

We can define, instead of $\tilde{\sigma}, \vec{n}$, the fields $s, \vec{\xi}$, with

$$\Sigma = \tilde{s}\, e^{i \vec{\sigma} \cdot \vec{\xi}/v} = \tilde{s} \cdot U(\vec{\xi})$$

Then $U = \cos\left(\frac{|\xi|}{v}\right) + i \frac{\vec{\sigma} \cdot \vec{\xi}}{\xi} \sin\left(\frac{|\xi|}{v}\right) \qquad \xi = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$

So $\tilde{\sigma} + i \vec{\sigma} \cdot \vec{n} = \Sigma = \tilde{s} \cos\left(\frac{\xi}{v}\right) + i \frac{\vec{\sigma} \cdot \vec{\xi}}{\xi} \sin\left(\xi/v\right)$

$$\Rightarrow \quad \tilde{\sigma} = s \cos(\xi/v), \quad n^i = \frac{\xi^i}{\xi} \sin(\xi/v)$$

This is a pretty non-linear transformation of the fields.

Now $\tilde{s}^2 = \frac{1}{2} \operatorname{Tr}(\Sigma^+ \Sigma) = \tilde{\sigma}^2 + \vec{n}^2$

and hence $\tilde{s}^2$ is invariant under $SU(2)_L \otimes SU(2)_R$. This

means that $\cancel{\operatorname{Tr}(\Sigma^+ \Sigma) \to \operatorname{Tr}(U_R^+ \Sigma \Sigma U_L^+) = U_R\, \tilde{s}\, U(\xi) U_L^+}$

$$U(\xi) \to U_L\, U(\xi)\, U_R^+ \quad , \text{ like } \Sigma$$

We, then, need to shift to $s = \tilde{s} - v$, such that the vacuum

is at $\langle \tilde{s} \rangle = v \Rightarrow \langle s \rangle = 0$.

The Lagrangian then is

$$\mathcal{L} = \bar{\Psi}_L i \not{\partial} \Psi_L + \bar{\Psi}_R i \not{\partial} \Psi_R + \frac{1}{4}\, \mathrm{tr}\left(\partial_\mu U^\dagger \partial^\mu U\right) \cdot (s+v)^2 + \frac{1}{2}\left((\partial_\mu s)^2 - 2\lambda v^2 s^2\right)$$
$$- \lambda v\, s^3 - \frac{\lambda}{4} s^4 + g\,(v+s)\left(\bar{\Psi}_L U \Psi_R + \bar{\Psi}_R U^\dagger \Psi_L\right)$$

To derive the Feynman rule for pion scattering we have to expand

$\frac{1}{4}(s+v)^2\, \mathrm{tr}(\partial_\mu U^\dagger \partial^\mu U)$ in terms of the $\vec{\xi}$ fields. One can show

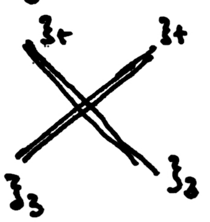
that $$\frac{1}{4}(s+v)^2\, \mathrm{tr}(\partial_\mu U^\dagger \partial^\mu U) = \frac{(v+s)^2}{4} \frac{2}{v^2}\left[(\partial_\mu \xi)^2 + \frac{\sin^2(\xi/v)}{(\xi/v)^2}\left[(\partial_\mu \vec{\xi})(\partial^\mu \vec{\xi}) - \ldots\right.\right.$$

$$= \frac{1}{2}\partial_\mu \vec{\xi}\, \partial^\mu \vec{\xi} + \frac{1}{6v^2}(\xi_i \partial_\mu \xi_i)(\xi_j \partial_\mu \xi_j) - \frac{1}{6v^2}\xi^2 \partial_\mu \vec{\xi} \cdot \partial^\mu \vec{\xi} + \frac{s}{v}\partial_\mu \vec{\xi}\, \partial^\mu \ldots$$

+ terms of higher dimensions.

It is not too hard to see that the two terms of order ξ^4 ~~contrib~~ combine

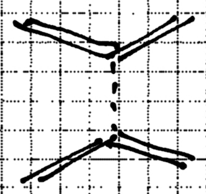
to give a Feynman rule for the $\xi_3 - \xi_3 - \xi_+ - \xi_+$ interaction equal

to  $\dfrac{(p_+ - p_+')^2}{4v^2}$

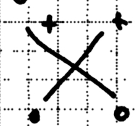
This is the same result we found in the linear representation of the Lagrangian for the σ-model. But note that in the non-linear representation above, there are no other interactions proportional to p^2 so the cancelations between diagrams, that was seen in the linear representation is not necessary here!

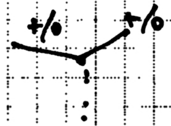
There is a theorem by Haag that guarantees that for every non-linear reparametrization of the fields $\phi = \xi \cdot F(\xi)$ with $F(0) = 1$, the classical observables are invariant.

Note that in the non-linear Lagrangian there are also $\frac{S}{U}\,\partial_\mu\vec{\xi}\,\partial^\mu\vec{\xi}$ interactio[ns]

but they would lead to a diagram  which would be

proportional to p^4 !

Actually, the ~~problem~~ feature of the linear Lagrangian that is responsible for the cancelation is that both the 4-vertex 

and the three-vertex 

are Goldstone boson ~~pion-pion or pion-sigma~~ interactions coming from terms in the linear $\mathcal{L}$ without derivatives. Derivative interactions bring momentum powers to the corresponding vertices. On the contrary, all Goldstone boson interactions in the non-linear Lagrangian are derivative interacti[ons] and as a result, the constant contribution cancelation is embedded automatically. This is a general property of Goldstone bosons, which is equivalent with the association of Goldstone bosons to ~~the~~ conserved currents of spontaneously broken symmetries.

Note the difference is the field content between linear & non-linear Lagrangian:

$\underbrace{\text{linear } \mathcal{L}}$ $\qquad\qquad$ $\underbrace{\text{non-linear } \mathcal{L}}$

$\left\{\begin{matrix}\hat{\sigma}\\ \vec{\pi}\end{matrix}\right\}$ transform as a bi-doublet $\qquad$ $\{\tilde{S}\}$ a singlet under $SU(2)\times S[U(2)]$

$\Sigma = \hat{\sigma}\mathbb{1} + i\vec{\pi}\cdot\vec{\sigma} \to U_L \Sigma U_R^\dagger$ $\qquad$ $\{\vec{\xi}\}$ transform non-linearly that is $U_L = U_R = U = e^{i\vec{\alpha}\cdot\vec{\sigma}}$, i.e. the isospin transformation.

It is actually the case that $\vec{\xi}$ transform linearly under the $SU(2)_V$ subgroup of $SU(2)\times SU(2)$ that is unbroken by the vacuum expectation value v^2.

Thus the G.B. transform non-linearly under the full, spontaneously broken, group, but they transform linearly under the un-broken subgroup of the system.

RECAP

$\boxed{\sigma\text{-model}}$ $\mathcal{L}$ invariant under $SU(2)_L \times SU(2)_R$, when $v^2 < 0$

$\boxed{v^2 > 0}$ $\longrightarrow$ $SU(2)_L \times SU(2)_R$] parent (broken) group.

$\downarrow$ breaks to

$SU(2)_V$ (isospin)] daughter group.

$\boxed{\text{Linear representation}}$	$\boxed{\text{Non-linear representation}}$
singlet under $SU(2)_V$ $\leftarrow$ $\hat{\sigma}$ becomes massive *	$\tilde{S}$ singlet under parent group!
triplet under $SU(2)_V$ $\leftarrow$ $\vec{\pi}$ massless G.B. **	$\vec{\xi}$ non-linear representation of parent group but triplet of the daughter group
ψ become massive	ψ as before
* CANNOT DECOUPLE $\hat{\sigma}$	* Can now decouple $\tilde{S}$ for low energy interactions.
** $\vec{\pi}$ and $\hat{\sigma}$ do not mix under $SU(2)_V$ but do mix under the general $\underbrace{\text{chiral } SU(2)_L\times SU(2)_R}_{\text{Parent group.}}$	

The non-linear σ-model is the leading term in a momentum expansion of the most general effective lagrangian with chiral symmetries