1. Spontaneous symmetry breaking crash course

Imagine a system described by a Lagrangian that is symmetric under a continuous transformation of the fields that contribute to it, whose physical states are not symmetric under the transformation. This is a situation where we say that the symmetry is spontaneously broken

example

$$\mathcal{L} = \frac{1}{2}\partial_\mu \vec{\varphi}\,\partial^\mu \vec{\varphi} - \frac{1}{2} m^2 \vec{\varphi}\cdot\vec{\varphi} - \frac{g}{4}(\vec{\varphi}\cdot\vec{\varphi})^2$$

where $\vec{\varphi} = \begin{pmatrix}\varphi_1\\ \vdots \\ \varphi_N\end{pmatrix}$ φ_i : real scalar fields.

The Lagrangian is invariant under $O(N)$ rotations of the fields (since it only depends on the rotationaly invariant $\vec{\varphi}\cdot\vec{\varphi}$). The potential is $V(\varphi) = \frac{1}{2} m^2 \vec{\varphi}\vec{\varphi} + \frac{g}{4}(\vec{\varphi}\cdot\vec{\varphi})^2$

if $m^2 < 0$ the minimum of the potential is not at $\vec{\varphi} = \vec{0}$, but at some $\vec{\varphi}_*$ such that $\vec{\varphi}_*\cdot\vec{\varphi}_* = -\frac{m^2}{g}$.

The "vacuum" configuration $\vec{\varphi}_*$ is not rotationally invariant, so if for example $\vec{\varphi}_* = \begin{pmatrix}\sqrt{-m^2/g}\\ 0 \\ \vdots\end{pmatrix}$, then any rotation that involves the first component of $\vec{\varphi}_*$ will send $\vec{\varphi}_* \to \vec{\varphi}_*'$.

The mass matrix in the Lagrangian is $(M^2)_{mn} = \left.\frac{\partial^2 V}{\partial\varphi_n \partial\varphi_m}\right|_{\vec{\varphi}=\vec{\varphi}_*}$

which can be seen to be equal to

$$(M^2)_{mn} = 2g\,\varphi_{*n}\varphi_{*m}$$

There is one eigenvector of M_{mn} with non-zero eigenvalue

$$m_0^2 = 2\varphi \; \vec{\varphi}_* \cdot \vec{\varphi}_* = 2\,|M^2|$$

and $N-1$ eigenvectors with zero eigenvalues.

Hence the theory contains one massive scalar and $N-1$ massless scalars.

Also observe that the theory has now $O(N-1)$ symmetry (rotations perpendicular to $\vec{\varphi}_*$ leave the vacuum invariant). There are $\frac{1}{2}N(N-1)$ generators of $O(N)$ and only $\frac{1}{2}(N-1)(N-2)$ of $O(N-1)$. So there are $\frac{1}{2}N(N-1) - \frac{1}{2}(N-1)(N-2) = N-1$ broken generators.

In general if the vacuum of a theory is breaking the symmetry of the Lagrangian, the theory contains one massless boson for every broken generator of the original symmetry group. These massless bosons are called (Nambu -) Goldstone bosons.

Note that the breaking of the symmetry is due to the quantum states of the system, not the operators. The current conservations are still valid $\qquad \partial_\mu \cdot J^\mu = 0 \qquad$ also for currents corresponding to broken generators.

It can be shown that the current J^μ associated with a broken symmetry has a non-zero probability to create a Goldstone boson out of the vacuum, i.e. that

$$\langle 0 |\; J^\mu(0) \;| G \rangle = f_G \cdot p^\mu$$

where p^μ is the momentum of the Goldstone boson. We don't prove this statement here (but see Weinberg, section 19.2).

Then the statement that the current is conserved is equivalent to the statement that the Goldstone boson is massless.

It also follows that any transition from a state $|a\rangle$ to a state $|b\rangle$ induced by the current J^μ will have a pole corresponding to an on-shell intermediate Goldstone boson.

$$\langle a|\, J^\mu(0)\, |b\rangle \sim \frac{i F q^\mu}{q^2} M_{ba} \qquad \text{where } q^2 = p_a^2 - p_b^2$$

2. Pions as Goldstone bosons

Let's go back to the chiral Lagrangian with isospin symmetry of ~~section~~ lecture 1:

$$\mathcal{L} = \bar\psi_u \not\partial \psi_u + \bar\psi_d \not\partial \psi_d = \bar\psi \begin{pmatrix} \not\partial & 0 \\ 0 & \not\partial \end{pmatrix} \psi$$

where $\psi_{u,d}$ are the Dirac fields for up and down quarks, considered massless, and $\psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$. We have seen that this Lagrangian is invariant under

$$\psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix} \longrightarrow e^{\,i \vec\theta\cdot\vec t + i\gamma_5 \vec\theta_A\cdot\vec t} \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$$

with $t_i = \frac{1}{2}\sigma_i$, σ_i being the Pauli matrices. We have seen that the conserved currents are $\vec J^\mu = \bar\psi \gamma^\mu \vec t \psi$ and $\bar\psi \gamma^\mu \gamma_5 \vec t \psi = \vec J_5^\mu$

We have reasons to believe this symmetry is broken:

(i) If the symmetry is exact, we expect for every hadron in the spectrum (made from up and down quarks), another hadron with equal quantum numbers and opposite parity. This is not realised!

(ii) Strong interactions can create bound states of quark-antiquark. These states must have zero momentum and angular momentum, so they automatically have negative parity. They would correspond to operators $\bar\psi_L \psi_R$ which break the axial isospin current.

But we have seen that the transition amplitude must have a pole at $q^2 \to 0$, so it follows that $g(q^2)$ has a pole, and hence

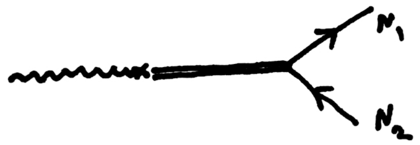
$$\lim_{q^2 \to 0} g(q^2) = \frac{2 m_N f(0)}{q^2}$$

So when $q^2 \to 0$

$$\langle N_1 | J_5^{\mu i} | N_2 \rangle \to \bar{u}_{N_1} \frac{q^\mu \gamma_5 \, 2 m_N f(0)}{q^2} u_{N_2} \qquad (*)$$

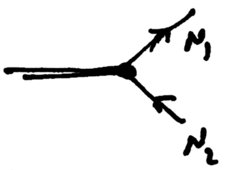
(all other terms are small compared to the $\frac{1}{q^2}$ enhanced one)

If we want to describe the process in a language that involves the pions directly, we would have, assuming an interaction term $2 i G_{\pi N} \vec{\pi} \bar{N} \gamma_5 \tau^i$

$$\langle N_1 | J_5^{\mu i} | N_2 \rangle \sim$$ 

where  $\to \langle 0 | J_5^{\mu i}(0) | \pi^j(p) \rangle = \frac{F_\pi}{2} \delta^{ij} p^\mu$

 $\to \frac{1}{p^2}$

 $\to +2 i G_{\pi N} \gamma_5$

So $$\langle N_1 | J_5^{\mu i} | N_2 \rangle \xrightarrow[q^2 = 0]{} \bar{u}_{N_1} \frac{q^\mu \gamma_5 G_{\pi N}}{q^2} u_{N_2} \qquad (**)$$

Comparing $(*)$ and $(**)$ we obtain the Goldberger–Treiman relation

$$G_{\pi N} = \frac{2 m_N f(0)}{F_\pi}$$

We can measure $m_N = \frac{m_p + m_n}{2} \approx 938.9$ MeV, $f(0) = 1.2527$, from low energy nuclear beta decays (that involve $J_5^{\mu i}$) and $F_\pi = 184$ MeV (from the pion decay – see exercise 1). This gives $G_{\pi N} \approx 12.7$, in fair agreement with the measured $G_{\pi N} = 13.5$

If we assume the $SU(2) \times SU(2)$ to be spontaneously broken we have

(i) three massless bosons in the theory, corresponding to the three broken $SU(2)$ generators

(ii) these Goldstone bosons can be generated from the vacuum by the three currents $J_5^{\mu i} = \bar{\psi} \gamma_\mu \gamma_5 t_i \psi$

Note that $J_5^{\mu 1}$ and $J_5^{\mu 2}$, involving quarks of different charge carry electric charge, while $J_5^{\mu 3}$ doesn't.

The three Goldstone bosons are identified as (massless) pions π^0, $\pi^\pm$

So we have $\langle 0 | J_5^{\mu i}(0) | \pi^j(p) \rangle = -i f_\pi \delta^{ij} p^\mu$

The currents $J^{\mu i}$ and $J_5^{\mu i}$ are exactly those that enter the weak interactions in the Standard Model. As a consequence, the pions decay to leptons plus neutrino, and we can measure f_π from the pion decay rate (see exercise 1).

(iii) the transition matrix between two nucleon states N_1, N_2, induced by $J_5^{\mu i}$ has a pole corresponding to the existence of the pions.

Generally $\langle N_1 | J_5^{\mu i}(0) | N_2 \rangle = \bar{u}_{N_1} \left[i\gamma^\mu \gamma_5 f(q^2) + q^\mu \gamma_5 g(q^2) + i [\gamma^\mu, \gamma^\nu] q_\nu \gamma_5 h(q^2) \right] u_{N_2}$

Current conservation means that $\bar{u}_{N_1} [i \not{q} \gamma_5 f(q^2) + q^2 \gamma_5 g(q^2)] u_{N_2} = 0$

Using $q = p_1 - p_2$ and $\bar{u}_{N_1} \not{p}_1 = m_{N_1}$ $\not{p}_2 u_2 = m_{N_2}$ we get

$$2 m_N f(q^2) = q^2 g(q^2) \quad (m_{N_1} = m_{N_2} \text{ here}).$$

3. Spontaneously broken symmetry and anomalies: the $\pi^0 \to \gamma\gamma$ decay.

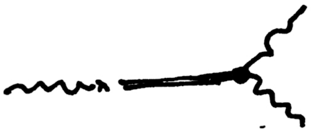
We proceed now to calculate the $\pi^0 \to \gamma\gamma$ decay rate. We begin from the matrix element

$$\langle p_1 p_2 | J_5^{\mu i} | 0 \rangle = \varepsilon_1^{*\lambda} \, \varepsilon_2^{*\nu} \quad M^{\mu\nu\lambda}(q^2)$$

where $\quad M^{\mu\nu\lambda}(q^2) = q^\mu \, \varepsilon^{\nu\lambda\alpha\beta} \, p_{1\alpha} \, p_{2\beta} \, M_1(q^2) + \ldots$

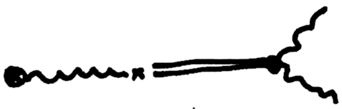
Since the symmetry is spontaneously broken, we expect pion contributions to the transition above, and it turns out that $M_1(q^2) = \dfrac{M}{q^2}$ + terms regular at $q^2 \to 0$. So, as $q^2 \to 0$, $M^{\mu\nu\lambda} \to q_\mu \varepsilon^{\nu\lambda\alpha\beta} p_{1\alpha} p_{2\beta} \cdot \dfrac{M}{q^2}$

The same transition in the language of pions would be described by



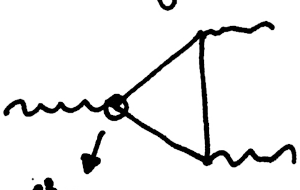
$$= i q^\mu f_\pi \, \frac{1}{q^2} \, A \, \varepsilon^{\nu\lambda\alpha\beta} \, p_{1\alpha} \, p_{2\beta}$$

To find what A is, we need to ~~compare~~ look at what is the equivalent of the conservation of current $J_5^{\mu i}$ in our case. The current $J_5^{\mu i}$ is not conserved in the presence of e/m fields, instead it contributes by an anomaly $\langle p_1 p_2 | \partial_\mu J_5^{\mu i} | 0 \rangle = B \; \varepsilon_1^\nu \, \varepsilon_2^\lambda \, \varepsilon_{\nu\lambda\alpha\beta} \, p_1^\alpha \, p_2^\beta$. Once we calculate B we can compare this to



$= i f_\pi A \, \varepsilon^{\nu\lambda\alpha\beta} \, p_{1\alpha} \, p_{2\beta} \, \varepsilon_1^\nu \varepsilon_2^\lambda$ from which we get that $A = \dfrac{B}{f_\pi}$.

The anomaly coefficient B is $\left(\dfrac{e^2}{16\pi^2}\right) \cdot \mathrm{tr}\{ Q^2 \cdot \sigma_3 \} \cdot N_c$



$Q^2 \sigma_3$

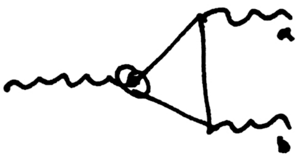
with $Q^2 = \begin{pmatrix} (2/3)^2 & 0 \\ 0 & (-1/3)^2 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

So we have $\dfrac{e^2}{16\pi^2} \quad \dfrac{N_c}{3}$

This leads to $A = \frac{e^2}{16\pi^2} \frac{N_c}{3} \cdot \frac{1}{f_\pi}$, which leads to a decay rate equal to (exercise 2) $\Gamma_{\pi_0 \to \gamma\gamma} = \left(\frac{N_c}{3}\right)^3 \cdot 1.11 \cdot 10^{16} s^{-1}$.

The observed rate is $1.19 \cdot 10^{16} s^{-1}$, in good agreement with the theory prediction if and only if $N_c = 3$!

Note that in pure QCD, the ABJ anomaly vanishes:



$$\sim \mathrm{Tr}[\sigma_i \, t_a t_b] = \mathrm{Tr}[\sigma_i] \, \mathrm{Tr}(t_a t_b) = 0$$

as a consequence of the fact that all quarks couple with the same charge to gluons!