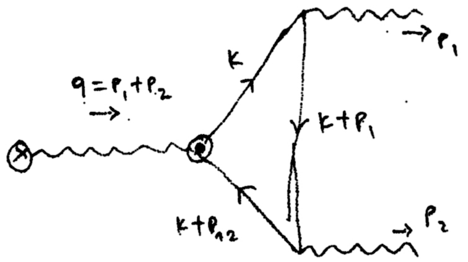
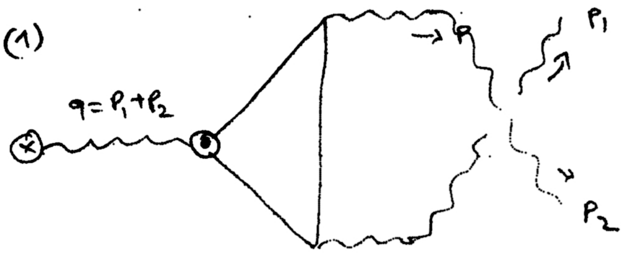


(2) (1)



$$q_\mu A_5^\mu(p_1,p_2) \quad + \quad q_\mu A_5^\mu(p_2,p_1)$$

$$q_\mu \cdot A_5^\mu(p_1,p_2) = \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}\left[\hat{k}\,\hat{q}\,\gamma_5(\hat{k}+\hat{p}_{12})\,\hat{\varepsilon}_2(\hat{k}+\hat{p}_1)\,\hat{\varepsilon}_1\right]}{k^2\,(k+p_{12})^2\,(k+p_1)^2}$$

$$q = p_1 + p_2 = k + p_{12} - k$$

$$\mathrm{Tr}\left[\hat{k}\left(\hat{k}+\hat{p}_{12}-\hat{k}\right)\gamma_5(\hat{k}+\hat{p}_{12})\,\hat{\varepsilon}_2(\hat{k}+\hat{p}_1)\,\hat{\varepsilon}_1\right]$$

$$= \mathrm{Tr}\left[\hat{k}(\hat{k}+\hat{p}_{12})\gamma_5(\hat{k}+\hat{p}_{12})\,\hat{\varepsilon}_2(\hat{k}+\hat{p}_1)\,\hat{\varepsilon}_1\right] - \mathrm{Tr}\left[\hat{k}^2\gamma_5(\hat{k}+\hat{p}_{12})\hat{\varepsilon}_2(\hat{k}+\hat{p}_1)\right.$$

$(*)$

$$= -\mathrm{Tr}\left[\hat{k}\,\gamma_5\,(k+p_{12})^2\,\hat{\varepsilon}_2(\hat{k}+\hat{p}_1)\,\hat{\varepsilon}_1\right] - \mathrm{Tr}\left[k^2\gamma_5(\hat{k}+\hat{p}_{12})\hat{\varepsilon}_2(\hat{k}+\hat{p}_1)\hat{\varepsilon}\right.$$

$$q_\mu A_5^\mu(p_1,p_2) = \int \frac{d^4k}{(2\pi)^4}(-)\frac{\mathrm{Tr}\left[\gamma_5\,\hat{\varepsilon}_2(\hat{k}+\hat{p}_1)\,\hat{\varepsilon}_1\hat{k}\right]}{k^2\,(k+p_1)^2} + \int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma_5\,\hat{\varepsilon}_1(\hat{k}+\hat{p}_{12})\hat{\varepsilon}_2(\hat{k}+\hat{p}_1\right.}{(k+p_{12})^2\,(k+p_1)^2}$$

$$= -\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma_5\,\hat{\varepsilon}_2(\hat{k}+\hat{p}_1)\,\hat{\varepsilon}_1\,\hat{k}\right]}{(k+p_1)^2\quad k^2} + \int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma_5\,\hat{\varepsilon}_1(\hat{k}+\hat{p}_2)\,\hat{\varepsilon}_2\,\hat{k}\right]}{(k+p_2)^2\quad k^2}$$

$$\Rightarrow q_\mu A_5^\mu(p_1,p_2) = -q_\mu A_5^\mu(p_2,p_1) \quad\Rightarrow\quad q_\mu\left[A_5^\mu(p_1,p_2) + A_5^\mu(p_2,p_1)\right] = 0$$

However, this is wrong result! The integrals are divergent and (i) we haven't used a regularization procedure which might alter the result and (ii) we have been shifting $k+p_1 \to k$ in an infinite integral. The proper way to calculate it is in Dimensional Regularization. But than $(*)$ is not correct: $(\hat{k}+\hat{p}_{12})\gamma_5 = -\gamma_5(\hat{k}+\hat{p}_{12}) + 2\gamma_5\hat{k}_\perp$

So $I_5 \equiv q\cdot A_\mu(p_1,p_2) + A_\mu(p_2\cdot p_1) = 2\int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{Tr}[\hat{k}\,\gamma_5\, k_\perp\, (\hat{k}+\hat{p}_{12})\,\hat{\varepsilon}_2\,(\hat{k}+\hat{p}_1)\cdot\hat{\varepsilon}_1]}{k^2\,(k+p_1)^2\,(k+p_{12})^2}$.

We need to compute this integral.

step 1. Employ Feyman parameters $\frac{1}{A\,B\,\Gamma} = \int \frac{dx\,dy\,dz\,\delta(1-x-y-z)}{(xA+yB+z\Gamma)^3}$.

$$I_5 = 2\int dx\,dy\,dz\,\delta(1-x-y-z)\int\frac{d^dk}{(2\pi)^d}\frac{\mathrm{Tr}[\qquad]}{\underbrace{[x k^2 + y(k+p_1)^2 + z(k+p_{12})^2]}^3}$$

$$k^2 + 2y\,k\cdot p_1 + 2z\,k\cdot p_{12} + z\,p_{12}^2$$
$$= (k+y p_1 + z p_{12})^2 - z^2 p_{12}^2 + z p_{12}^2 + 2yz\,p_1\cdot$$
$$= (k+Q)^2 - p_{12}^2\, z\,(1-z+2y)$$

with $Q = y p_1 + z p_{12}$.

step 2. Set $\ell = k+Q$

$$I_5 = 2\int dy\,dz \int\frac{d^d\ell}{(2\pi)^d}\frac{\mathrm{Tr}[(\hat{\ell}-\hat{Q})\,\gamma_5\,\ell_\perp\,(\hat{\ell}-\hat{Q}+\hat{p}_{12})\,\hat{\varepsilon}_2\,(\hat{\ell}-\hat{Q}+\hat{p}_1)\,\hat{\varepsilon}_1]}{[\ell^2-\Delta]^3}$$

step 3: Realize that due to the presence of $\ell_\perp$ only the terms $\sim\hat{\ell}^2$ survive. Those $\sim\hat{\ell}^3$ vanish because of integration and those $\sim\hat{\ell}^4$ because of the resulting contractions that leave traces with γ_5 and two γ-matrices.

note: $\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell_\perp^\mu\,\ell^\nu\,\ell^\rho\,\ell^\sigma}{(\ell^2-\Delta)^n} = \int\frac{d^d\ell}{(2\pi)^d}\frac{(\ell^2)^2}{D(D+2)}\left[g_\perp^{\mu\nu}g^{\rho\sigma} + g_\perp^{\mu\sigma}g^{\nu\rho} + g_\perp^{\mu\rho}g^{\nu\sigma}\right]$

Hence
$$I_5 = 2\int dy\,dz\,\underbrace{\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta)^3}\frac{(d-4)}{d}}\cdot\mathrm{Tr}[\gamma_5\hat{A}\,\hat{\varepsilon}_2\,\hat{B}\,\hat{\varepsilon}_1 + \gamma_5\hat{A}\,\hat{\varepsilon}_2\,\hat{\varepsilon}_1\,\hat{\Gamma} + \gamma_5\,\hat{\varepsilon}_2\,\hat{B}\,\hat{\varepsilon}_1$$

with $\hat{A} = -\hat{Q}+\hat{p}_{12}$, $\hat{B} = -\hat{Q}+\hat{p}_1$, $\hat{\Gamma} = -\hat{Q}$

$$\frac{d/2}{(4\pi)^{d/2}}\frac{\Gamma(2-\frac{d}{2})}{2}\,\Delta^{d/2-2}\,\frac{d-4}{d}$$
$$\parallel$$
$$\frac{\Delta^{-\varepsilon}}{(4\pi)^{2-\varepsilon}}\,\frac{1}{4}\,\Gamma(\varepsilon)\,(-2\varepsilon) = -\frac{1}{2}\frac{\Delta^{-\varepsilon}}{(4\pi)^{2\varepsilon}}\,\Gamma(1+\varepsilon).$$

Now, since the integral is actually finite, we can take the $\varepsilon \to 0$ limit to get

$$I_5 = \frac{e^2}{2\pi^2}\, \varepsilon^{\mu\nu\rho\sigma}\, p_{1\mu}\, \varepsilon_{1\nu}\, p_{2\rho}\, \varepsilon_{2\sigma} \; -$$

$$* \int d^4q \, e^{iqx} \, I_5 = \frac{e^2}{2\pi^2}\, \varepsilon^{\mu\nu\rho\sigma} \int d^4x \, e^{iqx} \left(p_{1\mu}\, \varepsilon_{1\nu}\, p_{2\rho}\, \varepsilon_{2\sigma}\right)$$

$$\langle p_1, p_2 | j_5^{\mu}(x) | 0 \rangle = \frac{e^2}{16\pi^2}\, \varepsilon^{\mu\nu\rho\sigma} \langle p_1, p_2 | F_{\mu\nu} F_{\rho\sigma} | 0 \rangle .$$

$$\Rightarrow \; j_5^{\mu}(x) = \frac{e^2}{16\pi^2}\, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \quad : \text{ the Adler - Bell - Jackiw anomaly}$$

* A path integral connection:

In the path integral formalism, Green's functions and scattering amplitudes are evaluated with the help of a path integral $\int [D\phi] \, e^{i\int d^4x \, \mathcal{L}(x)}$.

If a transformation leaves the Lagrangian invariant as well as the measure [D, it is a symmetry of the described system.

In the case of the axial trs. in the presence of a e/m field, i.e. for a Lagrangian $\bar{\psi} \not{D} \psi$, it turns out that $[D\psi][D\bar{\psi}]$ is not invariant under an (local) axial trs.

Instead $[D\psi]\,[D\bar{\psi}] \rightarrow e^{i\int d^4x \, a(x)\, \mathcal{A}(x)}$ with

$$\mathcal{A}(x) = -\frac{1}{16\pi^2}\, \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu}_{a}(x) \, F^{\rho\sigma}$$

So despite the fact that the Lagrangian is invariant, the path integral picks up a new term in the action, due to the trs. of the measure.

* Note that in order to derive Nöther's theorem for a global trs. in the path integral formalism one needs to start from a local trs. and vary the parameter field.

* Note that we cannot escape the anomaly while maintaining gauge invariance!