Inflation

The last topic in this course is related to spontaneous symmetry breaking of a scalar field in the very early universe, and the consequences of its acquiring a vacuum expectation value to the universe expansion.

Short introduction to the expansion of the Universe

Within the Standard Model of Cosmology the Universe originates from a Big Bang after which it expands with some, potentially non-constant rate. Isotropy and homogeneity are satisfied if the Universe is a manifold described by some metric of the Robertson-Walker type, where the line element (or proper time interval) satisfies

$$d\tau^2 = dt^2 - a^2(t)\left[d\vec{x}^2 + K\frac{(\vec{x}\cdot d\vec{x})^2}{1-K\vec{x}^2}\right]$$

or, in spherical coordinates

$$d\tau^2 = dt^2 - a^2(t)\left[r^2 d\underline{\Omega} + \frac{dr^2}{1-kr^2}\right]$$

with $d\underline{\Omega} = d\theta^2 + \sin^2\theta \, d\varphi^2$.

The constant K can be $-1, 0, 1$ corresponding to a hyperspherical, flat or spherical universe. The quantity $a(t)$ describes the radius of curvature of the universe.

In any such frame, the proper distance from $r=0$ to $r=R$ (with $dt=0$) is

$$D(R,t) = \int_0^R (d\tau^2)^{1/2} = \int_0^R \frac{a(t)\, dr}{\sqrt{1-kr^2}} = a(t)\, f(R,k)$$

with

$$f(R,k) = \begin{cases} \dfrac{R}{\sinh R} & k=-1 \\ R/\sin R & k=1 \\ R & k=0 \end{cases}$$

Hence $a(t)$ controls the proper distance between two points in the universe at any given time t.

The only unknowns in the RW metric, that determine the geometry are K and $a(t)$. They are determined by the matter-energy content of the Universe, according to Einstein's equations

$$R_{\mu\nu} = -8\pi G\, S_{\mu\nu}$$

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}_{\lambda}$$

$$R_{\mu\nu} = \partial_\nu \Gamma^{\lambda}_{\lambda\mu} - \partial_\lambda \Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma}$$

with $\Gamma^{\mu}_{\nu k} = \frac{1}{2} g^{\mu\lambda}\left[\partial_k g_{\lambda\nu} + \partial_\nu g_{\lambda k} - \partial_\lambda g_{\nu k}\right]$

It turns out that $R_{00} = 3\dfrac{\ddot{a}}{a}$

$$R_{ij} = \left[-2k + 2\dot{a}^2 + a\ddot{a}\right] g_{ij}$$

what is the energy-momentum tensor? Isotropy and homogeneity force $T^{00} = \rho(t)$ (space independent) and $T^{0i} = 0$, $T^{ij} = \tilde{g}^{ij}\dfrac{P(t)}{a^2(t)}$

Then energy-momentum conservation $T^{\mu\nu}{}_{;\nu} = 0$ for $\mu = 0$ becomes

$$\dot{\rho}(t) + 3\frac{\dot{a}(t)}{a(t)}\left(\rho(t) + P(t)\right) = 0$$

The Einstein eqs. become

$$3\frac{\ddot{a}}{a} = -4\pi G(3P + \rho)$$

$$-2\frac{\dot{a}^2}{a^2} - \frac{2K}{a^2} - \frac{\ddot{a}}{a} = -4\pi G(\rho - P)$$

$$\Rightarrow \boxed{\dot{a}^2(t) + K = \frac{8\pi G}{3}\rho(t)\, a^2(t)}$$

Remark 1 : If $k = 0, -1$ and $\rho(t) > 0$ then $\dot{a}(t) > 0$ always and the Universe never stops expanding!

Remark 2 : This equation, ~~that~~ can be written as

$$1 = \frac{8\pi G}{3} \frac{\rho(t)}{\dot{a}^2/a^2} - \frac{k}{a^2 \, \dot{a}^2/a^2}$$

or

$$1 = \frac{8\pi G}{3} \frac{\rho(t)}{H^2(t)} - \frac{k}{a^2 H^2(t)}$$

where $H(t) = \frac{\dot{a}(t)}{a(t)}$ is the Hubble constant that contols the Universe expansion. The equation holds for any time t, also for our time today, t_0, where we can measure $H_0 = H(t_0)$ from red shifts of distant objects. This all results in the fact that the matter density today determines what k is ($0, 1$ or -1).

Let's define the critical density today, $\rho_{crit} = \frac{3 H_0^2}{8\pi G}$, for which: if $\rho_0 = \rho_{crit}$ then $k = 0$, if $\rho_0 > \rho_{crit}$ $k = 1$ and if $\rho_0 < \rho_{crit}$ $k = -1$.

There are three potential sources that contribute to the density ρ_0: matter, energy in the form of radiation (i.e. photons) and a potential contribution from vaccuum expectation values of various fiel[ds]

We can define $\Omega_M = \frac{\rho_{M;0}}{\rho_{crit}}$, $\Omega_R = \frac{\rho_{R;0}}{\rho_{crit}}$, $\Omega_\Lambda = \frac{\rho_{\Lambda;0}}{\rho_{crit}}$

and formally $\Omega_k = -\frac{k}{a_0^2 H_0^2}$, upon which the Einstein eqs. take the simple form $1 = \Omega_M + \Omega_R + \Omega_\Lambda + \Omega_k$

Cold matter, radiation and the vaccuum density have different evolution with time, which is describe by their corresponding equation of state: $P = w \cdot \rho$

- For cold matter $P \simeq 0$ $(w=0)$
- For radiation $P = \rho/3$ $(w = 1/3)$
- For vacuum $P = -\rho$ $(w=-1)$.

The energy-momentum conservation $\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$ becomes

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho(1+w) = 0 \Rightarrow \frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \Rightarrow \rho(t) \sim a(t)^{-3(1+w)}$$

From which we see that
- cold matter $\rho(t) \sim a(t)^{-3}$
- radiation $\rho(t) \sim a(t)^{-4}$
- vacuum $\rho(t) = \text{const}.$

The three puzzles of the SM of cosmology

1. Flatness

As the Universe expands it passes from a period where radiation dominat to a period where matter dominates. Looking backwards in time, as $a(t) \to 0$, $\rho_M(t) \sim a^{-3}(t)$ $\rho_R(t) \sim a^{-4}(t)$ so in the Einstein eq.

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2$$ ~~relates~~ we can neglect k and get

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 \quad \text{or} \quad \rho = \frac{3H^2(t)}{8\pi G} = \rho_{cr}(t)$$

Note that this is the critical ~~energy~~ density at time t, not at time t_0 that we've seen before. If we can ignore k, then $a(t)$ is fully determined by $\dot{a}^2 \sim a^2 a^{-3(1+w)} \Rightarrow$ ~~$da\, a^{\frac{3(1+w)}{2}} \sim dt$~~

$$\Rightarrow \dot{a} \sim a^{(-1-3w)/2} \Rightarrow da\, a^{(1+3w)/2} \sim dt$$

$$\Rightarrow a^{3(1+w)/2} \sim t \Rightarrow a \sim t^{2/3(1+w)}$$

So during the matter dominated era $a \sim t^{2/3}$ while earlier, during the radiation dominated era $a \sim t^{1/2}$. ~~The same behaviour is found for $\dot{a}(t)$.~~ Then $\dot{a} \sim t^{-1/3}, t^{-1/2}$ respectively.

Today observations show that $|\Omega_k| < 1$ (compatible with $\Omega_k = 0$)

From the time of matter domination till today $\frac{-k}{\dot{a}^2} \sim \frac{-k}{t^{-2/3}} = -k\,t^{2/3}$

Earlier $\frac{-k}{\dot{a}^2} \sim \frac{-k}{(t^{-1/2})^2} = -k\,t$. In other words the curvature term has been increasing linearly from the begining of the radiation dominated period till $\sim 10^4$ K, and then also increasing as $t^{2/3}$ to arrive at todays very small value. In the begining of the radiation dominated era it must have been as small as 10^{-16}! Why?

This is the flatness problem. It can be solved by just assuming $k=0$. But if $k=1$ then the original curvature must have been extremely "fine-tuned".

2. The horizon problem

There are points in the Universe within our event horizon but outside each others event horizon. Such points can never have communicated with each other, so there is nothing that could have imposed thermal equilibr to them at the point of decoupling of photons from the plasma (300.000 ye after the Big Bang). Still we see that the Cosmic Background Radiatio from such disconnected points is extremely isotropic!

3. Monopoles from GUTs

If a GUT theory was based on a simple $SU(N)$ group that broke spontaneously to $SU(3) \times SU(2) \times U(1)$, the phase transition could have left the scalar field responsible for the spontaneous symmetry breaking ~~[illegible]~~ in a monopole configuration. Monopoles in causally disconnected regions could not have found each other to annihil leading to a huge relic abundance of such monopoles, of the order of one monopole per nucleon!

Inflation

The solution to all this problems (mainly the horizon one) that was proposed by Alan Guth and later improved by A. Linde and Albrecht and Steinhardt, is known as inflation. The idea is that the entire universe was in a much smaller space in the first fractions of a minute after the Big Bang, a region that was causally connected. However, for a short, 10^{-32} s period within the first minute, the Univers underwent an exponential growth (inflation) reaching the size necessary for the early radiation dominated period.

The exponential growth is driven by a scalar field, the inflaton, that lives in a potential which develops a secondary minimum as the tempera drops.

Scalar Field in Curved spacetime

The action in curved space-time becomes

$$I = \int d^4x \sqrt{-\det(g)} \left[\frac{1}{2} g_{\mu\nu} \partial^\mu \varphi \partial^\nu \varphi + V(\varphi) \right]$$

This ~~[illegible]~~ functional is stationary with respect to variations of the metric and the equivalent of Euler-Lagrange equations are now the energy-moment conservation,

$$T^{\mu\nu}{}_{;\nu} = \partial_\nu T^{\mu\nu} + \Gamma^\mu_{\kappa\nu} T^{\kappa\nu} + \Gamma^\nu_{\kappa\nu} T^{\kappa\mu} = 0$$

Performing such a variation on the action one can get the energy-moment tensor $T^{\mu\nu}$ and from there the expressions for density and pressure

$$\rho = +\frac{1}{2} g_{\mu\nu}\, \partial^\mu \varphi\, \partial^\nu \varphi + V(\varphi)$$

$$P = +\frac{1}{2} g_{\mu\nu}\, \partial^\mu \varphi\, \partial^\nu \varphi - V(\varphi)$$

For a spatially homogeneous vacuum exp. value

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$$

$$P = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)$$

The energy conservation $\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$ becomes

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} + V'(\varphi) = 0 \qquad (*)$$

If the scalar field dominates the Universe, $\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho$

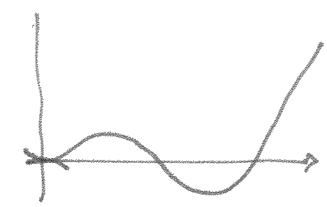
or $$\frac{\dot{a}(t)}{a(t)} \equiv H(t) = \sqrt{\frac{8\pi G}{3}\left[\dot{\varphi}^2/2 + V(\varphi)\right]}$$

or $$2H\dot{H} = \frac{8\pi G}{3}\left(\dot{\varphi}\ddot{\varphi} + V'(\varphi)\dot{\varphi}\right)$$

and using $(*)$ $$2H\dot{H} = \frac{8\pi G}{3}\dot{\varphi}\left(-3H\dot{\varphi} - V'(\varphi) + V'(\varphi)\right)$$

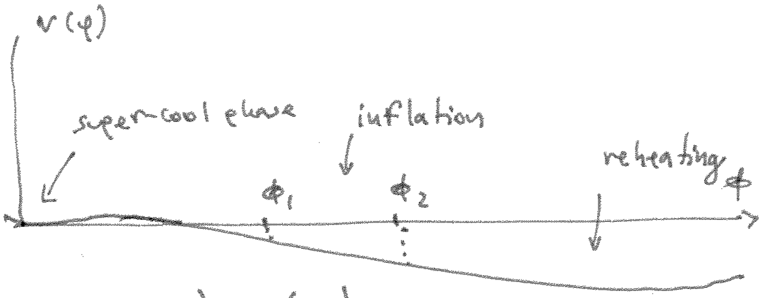
$$\Rightarrow \quad \dot{H} = -4\pi G\,\dot{\varphi}^2$$

→ The original inflation idea was that the inflaton potential develops at some critical temperature a non-zero minimum, but due to the potential barrier the phase transition is a first order one. The Universe is then in a super-cooled phase, and at some point leaks through the potential barrier towards a true minimum of the potential, ending the inflation period. However the phase-transition being first-order means that it happened through bubble nucleation, which in turn means all the energy would be concentrated on the bubble surface, i.e. the universe would have been very anisotropic.



The alternative, slow-roll inflation, assumes that the field's vev rolls down to the minimum slowly, through a potential barrier that is very low.

compared to the expansion



Under specific flatness conditions for the potential $V(\varphi)$ and its derivatives one can show that if the field shifts from φ_1 to φ_2 with $V(\varphi_1) > V(\varphi_2)$

then $\frac{\dot{a}(t)}{a(t)} = H(t) \Rightarrow \frac{a(t_2)}{a(t_1)} = \exp\left(\int_{t_1}^{t_2} H(t) \right) = \exp\left(\int_{\varphi_1}^{\varphi_2} \frac{H}{\dot{\varphi}} d\varphi \right)$

$$\approx \exp\left[- \int_{\varphi_1}^{\varphi_2} \frac{8\pi G\, V(\varphi)}{V'(\varphi)} d\varphi \right]$$

if $\left| \frac{V'(\varphi)}{V(\varphi)} \right| \ll \sqrt{16\pi G}$ then $\frac{a(t_2)}{a(t_1)} \gg e^{\sqrt{4\pi G}\,(\varphi_1 - \varphi_2)}$

The net effect is an exponential expansion fast enough to lead to all the observable universe originating from a tiny part within a nucleat. bubble.

Reheating

Once inflation ends, the potential energy of the inflaton, assuming that it interacts with other particles, is transfered to all the observable particles. The temperature returns to the pre-inflationary one