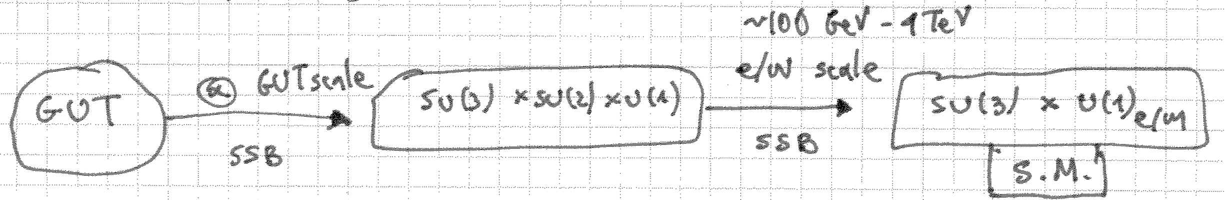
LECTURE 10

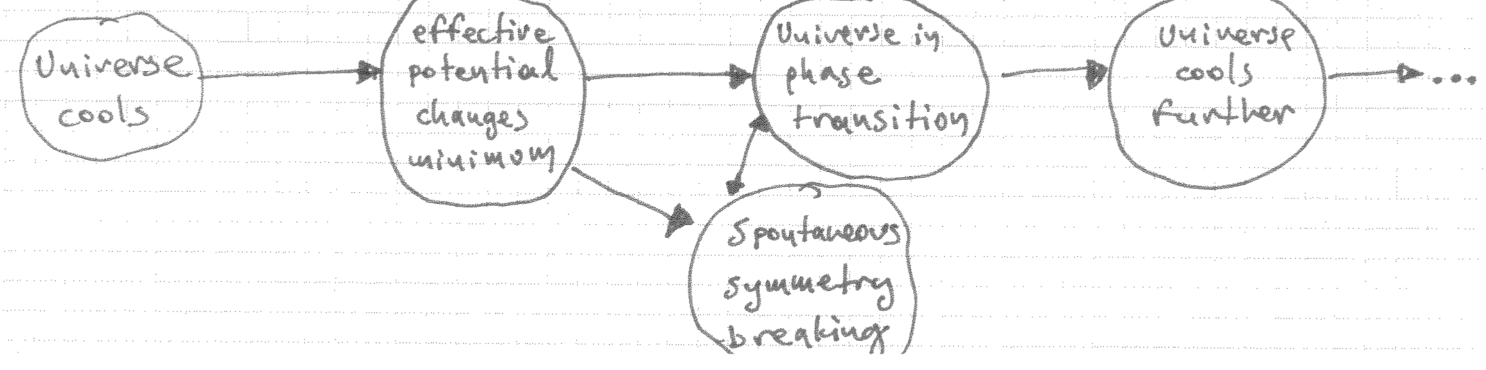
Phase transitions in the early universe

[following Bailin & Love]

The $SU(2)_L \times U(1)_Y$ gauge group of the SM is broken spontaneously to $U(1)_{e/m}$ by the Higgs boson acquiring a VEV. If particle interactions are further unified to a larger group (e.g. $SU(5)$) then at least one more spontaneous breaking should have occured from the GUT group to the SM $SU(3)_c \times SU(2)_L \times U(1)_Y$, presumably at some high scale called the GUT scale



In the early universe the particle density and the temperature (i.e. average energy) was immense. This resulted in an "effective potential" whose minimum different maybe than the minimum of zero temperature, could depend on the temperature, or equivalently, on the age of the universe. As the universe cools the effective potential approaches today's shape, and a succession of spontaneous symmetry breakings cascaded the symmetry group from the GUT group down to the SM one. As the minimum changes, the Universe undergoes a phase transition, from one minimum to the next, that could be of the first or second order, associated with each SSB.



During the phase transition the Universe picks one of potentially many vaccuum directions for a field to get a vev. This could have happened not in the same way throughout the universe, resulting in the formati of topological objects (like domain walls, cosmic strings, magnetic monopol like the Ferromagnet that picks different spin directions in different domains.

Non-uniform Spontaneous Symmetry Breaking $\longrightarrow$ formation of boundary regions with topological properties

The effective potential

In the path integral formalism we often introduce the "partition functional $Z[J] = \int D\phi \; e^{iS[\varphi] + i\phi \circ J}$

where $\quad a \circ b \equiv \int d^4x \; a(x)\, b(x)$

Then $\quad \dfrac{\delta Z}{\delta J(y)} = \int D\phi \; i\phi(y) \; e^{iS[\varphi] + i\phi\circ J}$

so $\quad \langle \phi(y) \rangle \equiv \dfrac{\int D\phi \; \phi(y) \; e^{iS[\varphi]}}{\int D\varphi \; e^{iS[\varphi]}} = \dfrac{-i}{Z} \left. \dfrac{\delta Z}{\delta J(y)} \right|_{J=0}$

We can define $\quad \phi_J(y) = -\dfrac{i}{Z} \dfrac{dZ}{dJ(y)} = -i \dfrac{d \log Z}{dJ(y)} = \dfrac{d\,W[J]}{\delta J(y)}$

where $\quad Z[J] = e^{iW[J]}$

Remember that $Z[J]$ is the sum of all vaccuum graphs, and thanks to the fact that connected graphs exponentiate, $W[J]$ is the sum of all connected graphs.

Let's denote, for every $\phi_J(x)$, a $J_\phi(x)$ such that

$$\phi_J(x) = \left.\frac{\delta W[J]}{\delta J_\phi(x)}\right|_{J=J_\phi}$$

We can then define

$$\Gamma[\phi_J] = -\phi_J \circ J_\phi + W[J_\phi]$$

(where it is implied that for every $\phi_J(x)$ there is a $J_\phi(x)$, so $W[J_\phi]$ is really a functional of ϕ_J). All this is just the generalization of the Legendre transform with J_ϕ at the role of the conjugate variable to $\phi_J(x)$.

Obviously
$$\frac{\delta \Gamma[\phi_J]}{\delta \phi_J(x)} = -J_\phi(x) - \phi_J \circ \frac{\delta J_\phi}{\delta \phi_J} + \frac{\delta W}{\delta \phi_J}$$

$$= -J_\phi(x) - \phi_J \circ \frac{\delta J_\phi}{\delta \phi_J} + \frac{\delta W}{\delta J_\phi} \circ \frac{\delta J_\phi}{\delta \phi_J}$$

$$= -J_\phi(x) - \phi_J \circ \frac{\delta J_\phi}{\delta \phi_J} + \phi_J \circ \frac{\delta J_\phi}{\delta \phi_J}$$

$$\Rightarrow \quad \frac{\delta \Gamma[\phi_J]}{\delta \phi_J(x)} = -J_\phi(x)$$

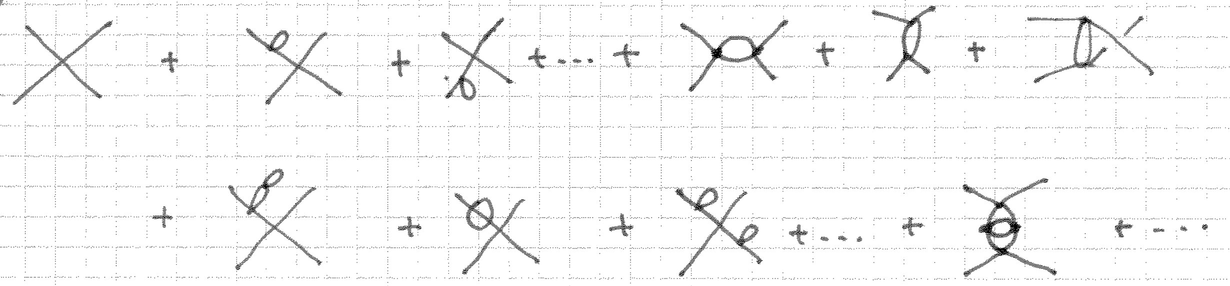
In particular, the fields configurations for which $\Gamma[\phi]$ is stationary, $\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = 0$ are those that correspond to $J_\phi(x)=0$ everywhere, i.e. they are vaccoum (exp. value) configurations: $\langle \phi \rangle = \left.\frac{dW}{dJ}\right|_{J=0}$.

$\Gamma[\phi]$ is the effective ~~potential~~ action, the ~~potential~~ action that takes into account quantum corrections.

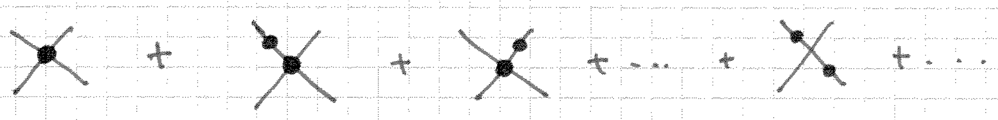
If we knew the effective action for a theory we could find the quantum-level eq. of motion, and solve them. But we don't know $\Gamma[\phi]$.

Moreover, $\Gamma[\phi]$ is the effective action in the sense that $W[J]$, the sum of connected graphs, can be calculated as the sum of connected tree graphs with vertices as if the action was $\Gamma[\phi]$ instead of $S[\phi]$!

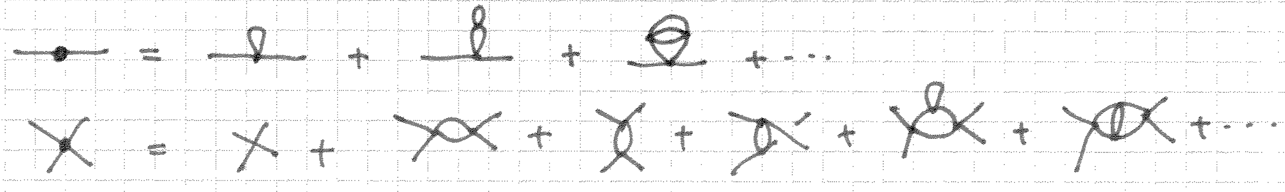
For example, in ϕ^4, a 4-point function would have the following connected ~~amplitude~~ graphs



which in terms of effective vertices would be



with



* $i\Gamma[\phi_0(x)]$ may be expressed as the sum of 1PI vacuum graphs calculated with a shifted action $S[\phi+\phi_0]$

$$i\Gamma[\phi_0] = \int_{\substack{1PI \\ \text{connected}}} D\phi \; e^{iS[\phi+\phi_0]}.$$

for a constant field ϕ_0

* The effective action is always proportional to the infinite space-time volume,

So $\Gamma[\phi_0] = -V_4\, V(\phi_0)$ where $V_4 = \int d^4x$

The action $$S[\phi+\phi_0] = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi+\phi_0)\right]$$
$$= \int d^4x \, \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi_0) - (\phi-\phi_0)V'(\phi_0) - \frac{1}{2}(\phi-\phi_0)^2 V''$$
$$\cdots$$
$$= -V(\phi_0)\int d^4x + \cdots$$

This term corresponds to the vaccuum graph of zero loop $(\bullet)$ and leads to the first term in the effective potential.

$$V_{eff}(\phi_0) = V(\phi_0) + \cdots$$

The next term in the potential comes from the one loop graph $(\bigcirc)$ corresponding to the quadradic terms in the action.

For $$S = \int d^4x \left[\lambda + \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi + \frac{1}{2}m^2\varphi^2 + \frac{g}{24}\varphi^4\right]$$

$$V_{eff}^{LO}(\phi_0) = \lambda + \frac{1}{2}m^2\phi_0^2 + \frac{g}{24}\phi_0^4$$

$$V_{eff}^{LO+NLO}(\phi_0) = \lambda_R + \frac{1}{2}m_R^2\phi_0^2 + \frac{g_R}{24}\phi_0^4 + \frac{\mu^4(\phi_0)\log\mu^2(\phi_0)}{64\pi^2}$$

where $$\mu^2(\phi) = m_R^2 + \frac{g_R\phi^2}{2}$$

To conclude, the effective potential equals the classical potential at LO, and then acquires corrections that induce renormalized constants in the potentia and change the potential dependence on ϕ_0.

The effective potential in finite temperature thermal field theory

We can see field theory as the limit of zero temperature of a thermal field theory, where asymptotic states are thought to be able to reach spatial and temporal infinity. In a finite temperature field theory scalar fields become periodic in $t = -i\tau$ with a period $b = \frac{1}{T}$

so $\phi(\tau, \vec{x}) = \phi(\tau + b, x)$ $\quad \tau = it$.

The partition function $Z = \int_{PBC} D\phi \; e^{-\int_0^b d\tau \int d^3\vec{x} \, \mathcal{L}(\varphi)}$

To derive the effective potential one has, as before, to evaluate the sum of 1PI, connected vacuum graphs with the action shifted from $\phi \to \phi + \phi_0$.

The leading order effective potential will again be the classical potential. The one-loop correction, however, will now be a path integral in thermal theory, involving the temperature through the upper limit of the τ-integration, $b = 1/T$.

In a general theory with scalars, fermions, and a $U(1)$ field, after shifting the scalar fields one gets quadradic terms from vers, that define the V_{eff}^{LO} (to be equal to $V(\phi)$) and a one loop correction that, in the limit of high temperature, becomes (when $T^2 \gg$ all masses)

$$V_{eff}^{LO+NLO} = V_{eff}^{LO+NLO}(T \to 0) - \frac{\pi^2 T^4}{90}\left(N_B + \frac{7}{8} N_F\right)$$

$$+ \frac{T^2}{24}\left[\operatorname{tr} M_S^2 + 3 \operatorname{tr} M_V^2 + 2 \operatorname{tr} M_F^2\right]$$

$$- \frac{T}{12\pi}\left[\left\{\operatorname{tr}(M_S^2)\right\}^{3/2} + 3\left[\operatorname{tr}(M_V^2)\right]^{3/2}\right] + \cdots$$

where M_S^2 is the mass matrix for the scalars after diagonalizati

and M_V^2, M_F^2 the corresponding matrices for the fermions and vector bosons.