Due by: 16 May

Exercise 1. ***Phase transitions in the Higgs model***

We will consider the Higgs model for a U(1) gauge field coupled to a complex scalar field, with Lagrangian [1]

$$\mathcal{L} = (D_\mu\varphi)(D^\mu\varphi^*) - m^2\varphi^*\varphi - \frac{\lambda}{4}(\varphi^*\varphi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1}$$

with usual covariant derivatives $D_\mu \equiv \partial_\mu + ieA_\mu$ and field strength tensor $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. Through spontaneous symmetry breaking, the scalar field acquires an vacuum expectation value (VEV), which we normalize for convenience as follows:

$$\langle\varphi\rangle = \frac{\varphi_c}{\sqrt{2}}. \tag{2}$$

The field can now be redefined around its VEV by the replacement:

$$\varphi \to \frac{1}{\sqrt{2}}(\varphi_c + \varphi_1 + i\varphi_2). \tag{3}$$

(a) Perform the substitution (3) in the Lagrangian (1) and read off the mass matrices (squared) M_S^2 and M_V^2 for the scalar and vector fields respectively.

(b) Compute the tree-level contribution to the effective potential $\bar{V}_0(\varphi_c)$ by substituting the VEV in the scalar potential.

In the lecture, you have shown that the 1-loop contribution to the effective potential is

$$\bar{V}_1(\varphi_c) = \bar{V}_1^{T=0}(\varphi_c) + \bar{V}_1^{T\neq 0}(\varphi_c), \tag{4}$$

with,

$$\bar{V}_1^{T=0}(\varphi_c) = B\varphi_c^4\left[\ln\frac{\varphi_c^2}{v^2} - \frac{25}{6}\right], \tag{5}$$

$$\bar{V}_1^{T\neq 0}(\varphi_c) = -\frac{\pi^2T^4}{90}(n_S + n_V) + \frac{T^2}{24}\left[t_S + 3t_V\right] - \frac{CT}{3}, \qquad (T \gg 0) \tag{6}$$

where we defined for convenience $B \equiv \frac{1}{64\pi^2}\left(\frac{5}{8}\lambda^2 + 3e^4\right)$, $v \equiv \sqrt{-4m^2/\lambda}$, $C \equiv \frac{1}{4\pi}\left(t_S^{3/2} + 3t_V^{3/2}\right)$ and $t_i = \operatorname{tr} M_i^2(\varphi_c)$ while $n_{S,V}$ denote the number of degrees of freedom from real scalar fields and vector fields.

The full 1-loop effective potential is then logically defined by $\bar{V}(\varphi_c) \equiv \bar{V}_0(\varphi_c) + \bar{V}_1(\varphi_c)$.

[1] We do not consider the gauge-fixing and ghost Lagrangians for the computation, so we simply drop them.

For the remaining of the exercise we will now consider the limiting case $e^4 \ll \lambda$.

(c) Compute $\bar{V}(\varphi_c)$ in this limit.

(d) Recast the relevant part of your result to obtain the "temperature-dependent mass (squared)" $m^2(T)$.

We will now analyze the various "phases" and compute their transition temperatures.

(e) Compute the temperature T_0 below which $\varphi_c = 0$ is no longer a local minimum of the the effective potential.

(f) Compute the temperature T_1 below which a second local minimum of the effective potential develops and show that $T_1 > T_0$.

(g) Show that the value of the VEV of the scalar field as a function of temperature can be written as

$$\varphi_c(T) = v \left[\frac{CT}{\sqrt{-\lambda m^2}} + \sqrt{1 - \frac{T^2}{T_1^2}} \right] , \tag{7}$$

and compute the mass of Higgs field and of the vector field.

Hint. The mass of a scalar field is related to the derivative of the potential at the VEV.