Due by: 2 May

Exercise 1. *Abelian anomaly in the path integral formalism*

In this exercise, we will consider the anomaly of a local chiral transformation in the path integral formalism.

The measure of the path integral for fermionic fields transforms as follows under a gauge transformation:

$$[d\psi][d\bar{\psi}] \to (\det \mathcal{U} \det \bar{\mathcal{U}})^{-1}[d\psi][d\bar{\psi}] \tag{1}$$

where ($\gamma_4 \equiv i\gamma_0$),

$$\mathcal{U}(x,y) \equiv U(x)\delta^{(4)}(x-y) \tag{2}$$

$$\bar{\mathcal{U}}(x,y) \equiv (\gamma_4 \otimes \mathbb{1})U^{\dagger}(x)(\gamma_4 \otimes \mathbb{1})\delta^{(4)}(x-y) \tag{3}$$

and $U(x) = D \otimes F$ is a matrix in with Dirac and flavour indices.

(a) Show that for a unitary local non-chiral transformation

$$U(x) = \exp\left[i\alpha(x)(\mathbb{1} \otimes t)\right] \tag{4}$$

where t is Hermitian, $\mathcal{U}$ is pseudo-unitary, i.e. $\int d^4y \bar{\mathcal{U}}(x,y)\mathcal{U}(y,z) = \mathbb{1}\delta^{(4)}(x-z)$.

(b) Show that for a unitary local chiral transformation

$$U(x) = \exp\left[i\alpha(x)(\gamma_5 \otimes t)\right] \tag{5}$$

where t is Hermitian, $\mathcal{U}$ is pseudo-hermitian, i.e. $\bar{\mathcal{U}}(x,y) = \mathcal{U}(x,y)$.

(c) We now focus on infinitesimal local chiral transformations. Using the property that $\det A = \exp \operatorname{tr} \ln M$ and the Taylor expansion of the logarithm, show that

$$(\det \mathcal{U})^{-2} = \exp\left[i\int d^4x \alpha(x)\mathcal{A}(x)\right] \tag{6}$$

with

$$\mathcal{A}(x) = -2\operatorname{tr}\left[\gamma_5 \otimes t\right]\delta^{(4)}(x-x). \tag{7}$$

The "$\delta^{(4)}(x-x)$" comes about by taking the analogue of the trace (summing over the diagonal) for the continuous variable: integrating over the "diagonal" $x = y$.

This expression needs to be regularized. We achieve that by replacing $\mathcal{A}$ with

$$\mathcal{A}(x) = \lim_{y\to x} -2\operatorname{tr}\left[(\gamma_5 \otimes t)f(-\not{D}^2/M^2)\right]\delta^{(4)}(x-y) \tag{8}$$

where $D_\mu = \partial_\mu \otimes \mathbb{1} - i(\mathbb{1} \otimes t^a)A^a_\mu$, M is the regulator (to be taken to infinity at the end) and f is such that,

$$f(0) = 1, \qquad f(\infty) = 0, \qquad sf'(s) = 0 \text{ for } s = 0, \infty. \tag{9}$$

(d) Propose functions satisfying (9).

(e) Use the Fourier representation of the δ-function and a clever rescaling to show that you can bring (8) into

$$\mathcal{A}(x) = -2M^4 \int \frac{d^4k}{(2\pi)^4} \operatorname{tr}\left[(\gamma_5 \otimes t) f\left(-(i\not{k} + \not{D}/M)^2\right)\right] \tag{10}$$

(f) Expanding the argument of f and performing a formal Taylor expansion of f around k^2, show that one can bring it further to the form,

$$\mathcal{A}(x) = -\int \frac{d^4k}{(2\pi)^4} f''(k^2) \operatorname{tr}\left[(\gamma_5 \otimes t)\not{D}^4\right] + \mathcal{O}(1/M) \tag{11}$$

Hint. Count the powers of M and the number of γ-matrices you need to get there and argue why the other terms can be ignored/vanish.

(g) Compute the integral over k (Minkowski!) using the properties (9). You should obtain,

$$\int d^4k f''(k^2) = i\pi^2. \tag{12}$$

(h) Show that

$$\not{D}^2 = D^2 \mathbb{1} - \frac{1}{4} t^a F^a_{\mu\nu} [\gamma^\mu, \gamma^\nu]. \tag{13}$$

(i) Wrapping up and taking the (partial) trace over the Dirac indices, perform the last steps leading you to the *anomaly function*:

$$\mathcal{A}(x) = -\frac{1}{16\pi^2} \varepsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^b_{\rho\sigma}(x) \operatorname{tr}[t^a t^b t]. \tag{14}$$

For a analysis of the problem with Euclidian path integrals, you might want to read the original paper: K. Fujikawa, Phys. Rev. Lett. **42**, 1195-1198 (1979).

Solution. Weinberg II, Section 22.2 (pp. 362-365).

Exercise 2. ***Magnetic monopoles***

Consider the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} + \frac{1}{2}D_\mu\varphi D^\mu\varphi - \frac{\lambda}{8}\left(\varphi^2 - \eta^2\right)^2 \tag{15}$$

where φ is a scalar field in the three-dimensional representation of $SO(3)$, the covariant derivative is given by

$$D_\mu\varphi_a = \partial_\mu\varphi_a - g\varepsilon_{abc}A^b_\mu\varphi_c \tag{16}$$

and the field strength tensor is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\varepsilon_{abc}A^b_\mu A^c_\nu \tag{17}$$

The magnetic monopole solution can be parametrised by the ansatz

$$\varphi_a = \frac{H(\xi)}{\xi}\eta\frac{x_a}{r} \tag{18}$$

$$A^a_i = \frac{\varepsilon_{aij}x_j}{gr^2}\left(K(\xi) - 1\right) \tag{19}$$

where $\xi = g\eta r$.

(a) Show that the covariant derivative is given in terms of this ansatz by

$$D_i\varphi_a = \frac{K(\xi)H(\xi)}{gr^4}\left(r^2\delta_{ai} - x_ax_i\right) + \left(\xi H'(\xi) - H(\xi)\right)\frac{x_ax_i}{gr^4} \tag{20}$$

(b) Show that the energy of the magnetic monopole is then given by

$$E = \frac{4\pi\eta}{g}\int_0^\infty d\xi\frac{1}{\xi^2}\left[\frac{1}{2}(\xi H' - H)^2 + H^2K^2 + (\xi K')^2\right. \tag{21}$$

$$\left. + \frac{1}{2}(K^2 - 1)^2 + \frac{\lambda}{8g^2}(H^2 - \xi^2)^2\right] \tag{22}$$

(c) Show that this energy is minimised for

$$\xi^2K'' = KH^2 + K(K^2 - 1) \tag{23}$$

$$\xi^2H'' = 2K^2H + \frac{\lambda}{2g^2}H(H^2 - \xi^2) \tag{24}$$

Solution. We insert the ansatz

$$\varphi_a = \frac{H(\xi)}{\xi}\eta\frac{x_a}{r}, \qquad A^a_i = \frac{\varepsilon_{aij}x_j}{gr^2}\left(K(\xi) - 1\right) \tag{S.1}$$

(a) We remark

$$\partial_i\frac{x_a}{r} = \frac{\delta_{ai}}{r} - \frac{x_ix_a}{r^3}, \qquad \partial_i\xi = g\eta\frac{x_i}{r} = g^2\eta^2\frac{x_i}{\xi}, \qquad \partial_ir^n = nx_ir^{n-2}. \tag{S.2}$$

We calculate the parts in order:

$$\partial_i\varphi_a = \partial_i\left(\frac{H}{\xi}\eta\frac{x_a}{r}\right) \tag{S.3}$$

$$= \eta\left(\frac{x_ix_a}{r^3}\xi\left[\frac{H'}{\xi} - \frac{H}{\xi^2}\right] + \frac{H}{\xi}\left[\frac{\delta_{ai}}{r} - \frac{x_ix_a}{r^3}\right]\right) \tag{S.4}$$

$$= \eta\left(\frac{x_ix_a}{r^3}\left[H' - 2\frac{H}{\xi}\right] + \frac{\delta_{ai}}{r}\frac{H}{\xi}\right) \tag{S.5}$$

$$-g\varepsilon_{abc}A_i^b\varphi_c = -g\varepsilon_{abc}\frac{\varepsilon_{bik}x_k}{gr^2}(K-1)\frac{H}{\xi}\eta\frac{x_c}{r} \tag{S.6}$$

$$= -\eta\frac{H}{\xi}\frac{1}{r^3}(K-1)(x_i x_a - r^2\delta_{ai}) \tag{S.7}$$

$$= \eta\left(\frac{x_i x_a}{r^3}\left[-\frac{H}{\xi}(K-1)\right] + \frac{\delta_{ai}}{r}\frac{H}{\xi}(K-1)\right) \tag{S.8}$$

adding up to

$$\partial_i\varphi_a - g\varepsilon_{abc}A_i^b\varphi_c = \eta\left(\frac{x_i x_a}{r^3}\left[H' - \frac{H}{\xi} - \frac{HK}{\xi}\right] + \frac{\delta_{ai}}{r}\frac{H}{\xi}K\right) \tag{S.9}$$

$$= KH\frac{1}{gr^4}\left(r^2\delta_{ai} - x_i x_a\right) + \frac{x_i x_a}{gr^4}\left(\xi H' - H\right). \tag{S.10}$$

(b) For the square we need

$$(r^2\delta_{ai} - x_i x_a)(r^2\delta_{ai} - x_i x_a) = r^4\delta_{ii} + x_i x_i x_a x_a - 2r^2 x_i x_i = 2r^4 \tag{S.11}$$

$$(r^3\delta_{ai} - x_i x_a)x_i x_a = r^4 - r^4 = 0 \tag{S.12}$$

and therefore we have for the kinetic term

$$\frac{1}{2}(D_\mu\varphi)(D^\mu\varphi) = \frac{-1}{2}(D_i\varphi)(D_i\varphi) = \frac{-1}{2}\frac{g^2\eta^4}{\xi^4}\left(2K^2H^2 + (\xi H' - H)^2\right). \tag{S.13}$$

We proceed with the field strength term, we have

$$\partial_i A_j^a = \frac{1}{g}\varepsilon_{ajk}\partial_i\left(x_k\frac{1}{r^2}(K-1)\right) \tag{S.14}$$

$$= \frac{1}{g}\varepsilon_{ajk}\left(\delta_{ik}r^{-2}(K-1) - 2x_i x_k r^{-4}(K-1) + x_k r^{-2}K'g\eta\frac{x_i}{r}\right) \tag{S.15}$$

$$= \frac{1}{g}\varepsilon_{aji}\frac{K-1}{r^2} + \frac{1}{g}x_i\varepsilon_{ajk}x_k\left(-2r^{-4}(K-1) + g\eta r^{-3}K'\right) \tag{S.16}$$

$$= \frac{-1}{g}\frac{K-1}{r^2}\varepsilon_{aij} + \frac{1}{g}x_i\varepsilon_{ajk}x_k r^{-4}\left(\xi K' - 2(K-1)\right) \tag{S.17}$$

and therefore

$$\partial_i A_j^a - \partial_j A_i^a = \frac{-2}{g}\frac{K-1}{r^2}\varepsilon_{aij} + \frac{1}{g}(x_i\varepsilon_{ajk}x_k - x_j\varepsilon_{aik}x_k)r^{-4}(\xi K' - 2(K-1)). \tag{S.18}$$

For the non-abelian part we have

$$\varepsilon_{abc}A_i^b A_j^c = \varepsilon_{abc}\varepsilon_{bik}x_k\varepsilon_{cjm}x_m\frac{1}{g^2}\frac{1}{r^4}(K-1)^2 \tag{S.19}$$

$$= (\delta_{ci}\delta_{ak} - \delta_{ck}\delta_{ai})x_k\varepsilon_{cjm}x_m\frac{1}{g^2}\frac{1}{r^4}(K-1)^2 \tag{S.20}$$

$$= x_a\varepsilon_{ijm}x_m\frac{1}{g^2}\frac{1}{r^4}(K-1)^2 \tag{S.21}$$

giving us

$$F_{ij}^a = \frac{-2}{g}\frac{K-1}{r^2}\varepsilon_{aij} + \frac{1}{g}(x_i\varepsilon_{ajk}x_k - x_j\varepsilon_{aim}x_m)r^{-4}\left(\xi K' - 2(K-1)\right) - \frac{1}{g}\frac{(K-1)^2}{r^4}x_a\varepsilon_{ijm}x_m. \tag{S.22}$$

In preparation for the contraction, we calculate

$$\varepsilon_{aij}\varepsilon_{aij} = 6 \qquad x_i\varepsilon_{ajk}x_k x_i\varepsilon_{ajm}x_m = 2r^4 \tag{S.23}$$

$$x_i\varepsilon_{ajk}x_k x_j\varepsilon_{ajm}x_m = 0 \qquad (x_i\varepsilon_{ajk}x_k - x_j\varepsilon_{aik}x_k)^2 = 4r^4 \tag{S.24}$$

$$x_a x_a\varepsilon_{ijm}\varepsilon_{ijk}x_m x_k = 2r^4 \qquad \varepsilon_{aij}(x_i\varepsilon_{ajk}x_k - x_j\varepsilon_{aik}x_k) = -4r^2 \tag{S.25}$$

$$\varepsilon_{aij}x_a\varepsilon_{ijm}x_m = 2r^2 \qquad (x_i\varepsilon_{ajk}x_k - x_j\varepsilon_{aik}x_k)x_a\varepsilon_{ijm}x_m = 0 \tag{S.26}$$

We insert this into $F_{ij}^a F^{ij,\,a} = F_{ij}^a F_{ij}^a = (F_{ij}^a)^2$:

$$(F_{ij}^a)^2 = \frac{24}{g^2}\frac{(K-1)^2}{r^4} + \frac{1}{g^2}4r^{-4}(\xi K' - 2(K-1)^2)^2 + \frac{1}{g^2}\frac{(K-1)^4}{r^8}\cdot 2r^4 \tag{S.27}$$

$$-\frac{4}{g^2}(K-1)(\xi K' - 2(K-1))(-4r^2)r^{-6} + \frac{4}{g^2}\frac{(K-1)^3}{r^6}\cdot 2r^2 \tag{S.28}$$

$$= \frac{1}{g^2r^4}\left[8(K-1)^2 + 2(K-1)^4 + 8(K-1)^3 + 4(\xi K')^2\right] \tag{S.29}$$

$$= \frac{1}{g^2r^4}\left[2(K^2-1)^2 + 4(\xi K')^2\right] \tag{S.30}$$

$$\frac{-1}{4}(F^a_{ij})^2 = \frac{-g^2\eta^4}{2\xi^4}\left[(K^2-1)^2 + 2(\xi K')^2\right] \tag{S.31}$$

We do need the potential term as well, we have

$$\frac{-\lambda}{8}\left(\varphi^2 - \eta^2\right)^2 = \frac{-\lambda}{8}\frac{\eta^4}{\xi^4}\left(H^2 - \xi^2\right)^2. \tag{S.32}$$

Now we add things up to arrive at the energy

$$E = -\int d\vec{x}\,\mathcal{L} = -\int \frac{4\pi}{g^3\eta^3}\xi^2 d\xi\,\mathcal{L} = \frac{4\pi}{g^3\eta^3}\int \xi^2 d\xi \left(\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} - \frac{1}{2}D_\mu\varphi D^\mu\varphi + \frac{\lambda}{8}\left(\varphi^2-\eta^2\right)^2\right) \tag{S.33}$$

$$= \frac{4\pi\eta}{g}\int d\xi \frac{1}{\xi^2}\left(\frac{1}{2}(K^2-1)^2 + (\xi K')^2 + K^2H^2 + \frac{1}{2}(\xi H' - H)^2 + \frac{\lambda}{8g^2}(H^2-\xi^2)^2\right). \tag{S.34}$$

(c) We assume $E[g] = \int f(g(x), g'(x))dx$ and we recall

$$\frac{\delta E}{\delta g}[\varphi] = \frac{d}{d\varepsilon}\int f(g+\varepsilon\varphi, g'+\varepsilon\varphi')dx\bigg|_{\varepsilon=0} \tag{S.35}$$

$$= \int \frac{\partial f(g,g')}{\partial g}\varphi + \frac{\partial f(g,g')}{\partial g'}\varphi' dx \tag{S.36}$$

$$= \int \left(\frac{\partial f(g,g')}{\partial g} - \frac{\partial}{\partial x}\frac{\partial f(g,g')}{\partial g'}\right)\varphi\, dx \tag{S.37}$$

where we have assumed that the boundary terms of the partial integration vanish. Therefore we get from the condition that the derivatives of the energy with respect to K and H should vanish:

$$\frac{\delta E}{\delta K} = 0 \Rightarrow K'' = \frac{1}{\xi^2}\left(K(K^2-1) + 2KH^2\right) \tag{S.38}$$

$$\frac{\delta E}{\delta H} = 0 \Rightarrow H'' - \frac{H'}{\xi} + \frac{H}{\xi^2} = \frac{2K^2H}{\xi^2} - \frac{H'}{\xi} + \frac{H}{\xi^2} + \frac{1}{\xi^2}\frac{\lambda}{2g^2}H(H^2-\xi^2). \tag{S.39}$$