Due by: 18 April

Exercise 1. ***Cartan-Maurer integral invariant I***

In order to study topological effects in quantum field theory, one needs to make sure that the topological quantities are defined in a appropriate manner.

Consider a compact n-dimensional manifold S with coordinates $x_1, \ldots, x_n$. Assuming we have a mapping $g : S \to G$ where G is a manifold of non-singular ($\det g \neq 0$) matrices, we define the *Cartan-Maurer form*,

$$I[g] = \int d^n x \, \varepsilon^{i_1 \cdots i_n} \operatorname{tr} \left[g^{-1}(\partial_{i_1} g) \cdots g^{-1}(\partial_{i_n} g) \right] , \tag{1}$$

where $\partial_k \equiv \frac{\partial}{\partial x^k}$ and g^{-1} denotes the matrix inverse of g ($g^{-1}g = gg^{-1} = \mathbb{1}$).

(a) What can you say about $I[g]$ without performing the calculation in the case where n is even?

(b) Show that $I[g]$ is invariant under an infinitesimal change of the mapping $g \to g + \delta g$:

(i) Argue why you can get the full variation only acting on the last factor.

(ii) Compute it and show that you can cast it as,

$$\delta \left(g^{-1}(\partial_{i_n} g) \right) = g^{-1} \partial_{i_n} (\delta g \, g^{-1}) g \tag{2}$$

Hint. Remember that you are dealing with matrices!

(iii) Perform integration by part and argue why the terms vanish or cancel so that, in the end, $\delta I[g] = 0$.

Having shown that $I[g]$ is invariant under small variations, we have also proved that if two mappings are continuously deformable into one another (i.e. they belong to the same homotopy class), they yield the same Cartan-Maurer invariant and thus the latter depends only on the homotopy class c to which g belongs, and we denote it by $I(c)$.

Taking two mappings $g_1, g_2 : S \to G$ (which represent their respective homotopy classes c_1 and c_2), one can define the concatenation,

$$g(x) = \begin{cases} g_1(2x_1, x_2, \ldots, x_n) & 0 \leq x_1 \leq \frac{1}{2} \\ g_2(2x_1 - 1, x_2, \ldots, x_n) & \frac{1}{2} \leq x_1 \leq 1 \end{cases} , \tag{3}$$

with homotopy class $c_1 \otimes c_2$.

(c) Show that the set of homotogy classes together with the concatenation $(C, \otimes)$ form a group by constructing maps corresonding to the neutral element e and the inverse c^{-1} of an element $c \in C$.

We call $(C, \otimes)$ the *homotopy group* and denote it by $\pi_n(G)$.

(d) Using (1), show that $I(c_1 \otimes c_2) = I(c_1) + I(c_2)$ and compute $I(e)$, $I(c^{-1})$ and $I(\underbrace{c \otimes \cdots \otimes c}_{n})$.

Exercise 2. ***Cartan-Maurer integral invariant II***

Keeping the setup of the previous exercise, one can specify to the case where G is a Lie (sub)group, one can define a metric on it as a function of the coordinates. For instance,

$$\gamma_{ij}(x) = -\frac{1}{2}\text{tr}\,\left[g^{-1}(\partial_i g)g^{-1}(\partial_j g)\right], \tag{4}$$

has the necessary properties. Furthermore, using the generators of G one can show that the Cartan-Maurer invariant can be written as,

$$I[g] = (2i)^n \varepsilon^{i_1 \cdots i_n}\text{tr}\,[t_{i_1}\cdots t_{i_n}]\frac{1}{\sqrt{\det\gamma(0)}}\int d^n x\sqrt{\det\gamma(x)}. \tag{5}$$

We will now specify by taking $S = S^3$ (hence $n = 3$) and $G = \text{SU}(2)$ and perform the explicit calculation. A judicious choice of the mapping is,

$$g(x) = x_4 + i\vec{x}\cdot\vec{\sigma}. \tag{6}$$

(a) Compute the metric (4) for this mapping.

(b) Finish the computation to show that $I[g] = 24\pi^2$.

Hint. Make use of the properties of the Pauli matrices.