Due by: 11 April

Exercise 1. *Topological charge*

Consider a SU(N) gauge theory, with the covariant derivative defined as

$$D_\mu \equiv \partial_\mu + igA_\mu, \quad A_\mu \equiv A^a_\mu T^a, \quad F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu \tag{1}$$

where the T^a are the generators of SU(N), satisfying $[T^a, T^b] = if^{abc}T^c$.

(a) Show that $F^a_{\mu\nu}\tilde{F}^{\mu\nu,a} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$ can be written as the total derivative of the *Cherns-Simons current*,

$$K^\mu = \varepsilon^{\mu\nu\rho\sigma}\left(A^a_\nu F^a_{\rho\sigma} + \frac{g}{3}f_{abc}A^a_\nu A^b_\rho A^c_\sigma\right) \tag{2}$$

The *topological charge* associated to K^μ is defined as

$$n \equiv \frac{g^2}{32\pi^2}\int d^3x\, K_0\Big|^{t=+\infty}_{t=-\infty} \tag{3}$$

Consider an adiabatic transformation

$$A_\mu(t=-\infty) = 0 \tag{4}$$

$$A_\mu(t=+\infty) = \frac{i}{g}\left(\partial_\mu\Lambda\right)\Lambda^{-1} \qquad \Lambda = \frac{\vec{x}^2 - d^2}{\vec{x}^2 + d^2}\mathbb{1} + i\frac{2d}{\vec{x}^2 + d^2}x_k\tau_k \tag{5}$$

where the τ_i denote as usual the generators of SU(2): $\tau_i\tau_j = \delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\tau_k$.

(b) Check that $F_{\mu\nu} = 0$, so that the topological charge reads

$$n = \frac{g^3}{96\pi^2}\int d^3x\ \varepsilon^{ijk}f^{abc}A^a_i A^b_j A^c_k = -i\frac{g^3}{24\pi^2}\int d^3x\ \varepsilon^{ijk}\mathrm{tr}\, A_i A_j A_k \tag{6}$$

Hint. Express f^{abc} from a trace to obtain the last equality.

(c) Show that

$$A_i(t=+\infty) = \frac{-2\,d}{g(\vec{x}^2 + d^2)^2}\left[(\vec{x}^2 - d^2)\tau_i - 2(x_j\tau_j)x_i + 2d\,\varepsilon_{ijk}x_j\tau_k\right] \tag{7}$$

(d) Compute the topological charge n of this adiabatic transformation.

Hint. Recall the contraction identities for Levi-Civita tensors.