Due by: 28 March

Exercise 1. ***Gell-Mann–Okubo Mass Formula and Weinberg Ratio of Quark Masses***

Start from the Lagrangian of chiral perturbation theory at order p^2 as stated in the lecture:

$$\mathcal{L}_{\chi\mathrm{PT},p^2} = \frac{v^2}{4}\mathrm{tr}\left(D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U\right), \tag{1}$$

$$U = \exp(i\sqrt{2}\Phi/v), \qquad \Phi = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -2\frac{\eta_8}{\sqrt{6}}. \end{pmatrix}. \tag{2}$$

Calculate the masses of the particles by inserting $D_\mu = \partial_\mu$, $\chi = 2BM$, $M = \mathrm{diag}(m_u, m_d, m_s)$ and expanding the Lagrangian up to the second order in Φ. Verify the Gell-Mann–Okubo mass formula

$$4m_K^2 - 3m_\eta^2 - m_\pi^2 = 0 \tag{3}$$

and the Weinberg ratio of quark masses

$$\frac{2m_K^2 - m_\pi^2}{m_\pi^2} = \frac{2m_s}{m_d + m_u} \tag{4}$$

where $m_\pi^2 \equiv \frac{1}{3}(m_{\pi^+}^2 + m_{\pi^-}^2 + m_{\pi^0}^2)$, $m_K^2 \equiv \frac{1}{4}(m_{K^0}^2 + m_{\overline{K}^0}^2 + m_{K^-}^2 + m_{K^+}^2)$.

Solution. We expand up to second order in Φ:

$$U \approx 1 + i\frac{\sqrt{2}}{v}\Phi - \frac{2}{v^2}\Phi^2 \tag{S.1}$$

which gives us the Lagrangian (inserting $D_\mu = \partial_\mu$ and $\chi = 2BM$ as well)

$$\frac{v^2}{4}\mathrm{tr}\left(\frac{2}{v^2}\partial_\mu\Phi\partial^\mu\Phi + 2BM\left(1 - i\frac{\sqrt{2}}{v}\Phi - \frac{2}{v^2}\Phi^2\right) + 2BM\left(1 + i\frac{\sqrt{2}}{v}\Phi - \frac{2}{v^2}\Phi^2\right)\right) \tag{S.2}$$

we omit a constant term and have

$$\mathcal{L} = \frac{1}{2}\mathrm{tr}\left(\partial_\mu\Phi\partial^\mu\Phi\right) - \mathrm{tr}\left(2BM\Phi^2\right). \tag{S.3}$$

We can write this Lagrangian as a sum of Lagrangians for scalar and complex fields plus a pion eta interaction

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\pi^0)(\partial^\mu\pi^0) - \frac{1}{2}m_{\pi^0}^2(\pi^0)^2 + \frac{1}{2}(\partial_\mu\eta_8)(\partial^\mu\eta_8) - \frac{1}{2}m_{\eta_8}^2(\eta_8)^2 + (\partial_\mu\pi^+)(\partial^\mu\pi^-) - m_{\pi^+}^2\pi^+\pi^- \tag{S.4}$$

$$+ (\partial_\mu K^0)(\partial^\mu\overline{K}^0) - m_{K^0}^2 K^0\overline{K}^0 + (\partial_\mu K^+)(\partial^\mu K^-) - m_{K^+}^2 K^+K^- + \frac{2B}{\sqrt{3}}(m_d - m_u)(\pi^0\eta_8) \tag{S.5}$$

with the mass parameters

$$m_{\pi^0}^2 = 2B(m_d + m_u), \quad m_{\eta_8}^2 = \frac{2B}{3}(m_u + m_d + 4m_s) \tag{S.6}$$

$$m_{\pi^+}^2 = 2B(m_u + m_d), \quad m_{K^+}^2 = 2B(m_u + m_s), \quad m_{K^0}^2 = 2B(m_d + m_s) \tag{S.7}$$

which obey the Gell-Mann–Okubo relation and the Weinberg ratio of quark masses.

Exercise 2. ***Semileptonic tau decay***

We consider the partial width of the semileptonic decay of the tau: $\tau^+ \to \bar{\nu}_\tau \pi^+$. This process is related to $\pi^+(p) \to \ell^+(k)\nu_\ell(q)$ treated in exercise 3.

(a) Starting from

$$\overline{\left|\mathcal{M}_{\pi^+\to\ell^+\nu_\ell}\right|^2} = 8\, G_F^2 f_\pi^2 \left(2(q\cdot p)(k\cdot p) - p^2(q\cdot k)\right), \tag{5}$$

cross the lepton to the initial state, the pion to the final state, to show that

$$\Gamma = \frac{1}{8\pi} G_F^2 f_\pi^2 m_\ell^3 \left(1 - \frac{m_\pi^2}{m_\ell^2}\right)^2 . \tag{6}$$

(b) Is this process allowed for all lepton flavours? Why?

Solution.

(a) We cross $k \to -k$, $p \to -p$, adding an overall (-1) because we have crossed a fermion and a prefactor $1/2$ because we are now averaging over the spin of the incoming tau to arrive at

$$\overline{\left|\mathcal{M}_{\ell^-\to\pi^-\nu_\ell}\right|^2} = \frac{1}{2}\sum_{\text{spins}} \left|\mathcal{M}_{\ell^-\to\pi^-\nu_\ell}\right|^2 = 4G_F^2 f_\pi^2 \left(2(q\cdot p)(k\cdot p) - p^2(q\cdot k)\right). \tag{S.8}$$

We determine the scalar products from $k = p + q$, $k^2 = m_\ell^2$, $p^2 = m_\pi^2$ and $q^2 = 0$ as

$$p\cdot q = \frac{1}{2}(m_\ell^2 - m_\pi^2), \quad k\cdot q = \frac{1}{2}(m_\ell^2 - m_\pi^2), \quad k\cdot p = \frac{1}{2}(m_\ell^2 + m_\pi^2) \tag{S.9}$$

which we insert to have

$$\overline{|\mathcal{M}|^2} = 2G_F^2 f_\pi^2 m_\ell^4 \left(1 - \frac{m_\pi^2}{m_\ell^2}\right). \tag{S.10}$$

We combine this with the integrated two-particle phase space

$$\Phi_{1\to2} = \frac{1}{8\pi}\left(1 - \frac{m_\pi^2}{m_\ell^2}\right) \tag{S.11}$$

to arrive at

$$\Gamma = \frac{1}{2m_\ell}\overline{|\mathcal{M}|^2}\Phi_{1\to2} = \frac{1}{8\pi} G_F^2 f_\pi^2 m_\ell^3 \left(1 - \frac{m_\pi^2}{m_\ell^2}\right)^2 . \tag{S.12}$$

(b) No. Look at the masses!