Due by: 21 March

Exercise 1. ***Linear sigma model***

The linear sigma model consists of two Dirac fermions that we arrange in a vector $\psi = \begin{pmatrix} u \\ d \end{pmatrix}$ and four real scalars $\tilde{\sigma}$ and $\vec{\pi} = (\pi^1, \pi^2, \pi^3)$ descibed by the Lagrangian density

$$\mathcal{L}_\sigma^{\text{lin}} = \frac{1}{2}(\partial_\mu \tilde{\sigma})(\partial^\mu \tilde{\sigma}) + \frac{1}{2}(\partial_\mu \vec{\pi}) \cdot (\partial^\mu \vec{\pi}) - \frac{\lambda}{4}(\tilde{\sigma}^2 + \vec{\pi} \cdot \vec{\pi} - v^2)^2 + \bar{\psi} i \partial\!\!\!/ \psi + g\bar{\psi}(\tilde{\sigma} + i\vec{\sigma} \cdot \vec{\pi}\gamma_5)\psi. \quad (1)$$

(a) Show that (1) can be recast as

$$\mathcal{L}_\sigma^{\text{lin}} = \frac{1}{4}\text{tr}\,(\partial_\mu \Sigma)^\dagger (\partial^\mu \Sigma) - \frac{\lambda}{4}\left[\frac{1}{2}\text{tr}\,\Sigma^\dagger \Sigma - v^2\right]^2 + \bar{\psi}_L i\partial\!\!\!/ \psi_L + \bar{\psi}_R i\partial\!\!\!/ \psi_R + g\bar{\psi}_L \Sigma \psi_R + g\bar{\psi}_R \Sigma^\dagger \psi_L \quad (2)$$

where $\Sigma \equiv \tilde{\sigma}\mathbb{1} + i\vec{\sigma} \cdot \vec{\pi}$ and $\psi_{R,L}$ are the right-handed and left-handed parts.

The fields $\psi_{R,L}$ and Σ transform accrording to

$$\psi_{R,L} \to U_{L,R}\psi_{R,L}, \quad \Sigma \to U_L \Sigma U_R^\dagger, \qquad U_{R,L} \equiv \exp\left(-\frac{1}{2}\vec{\alpha}_{R,L} \cdot \vec{\sigma}\right) \in \text{SU}(2). \quad (3)$$

(b) Using the properties of the Pauli matrices, recover $\tilde{\sigma}$ and $\vec{\pi}$ from Σ and deduce their transformation properties.

Exercise 2. ***Non-linear sigma model***

(a) Using the properties of the Pauli matrices, prove that

$$U \equiv \exp\left(i\frac{\vec{\sigma} \cdot \vec{\xi}}{v}\right) = \cos\left(\frac{\xi}{v}\right) + i\frac{\vec{\sigma} \cdot \vec{\xi}}{\xi}\sin\left(\frac{\xi}{v}\right), \quad \xi = |\vec{\xi}|. \quad (4)$$

The Lagrangian of the non-linear sigma model contains the term

$$\mathcal{L}_\sigma^{\text{non-lin}} \supset \Delta\mathcal{L} = \frac{1}{4}(v + S)^2 \text{tr}\,(\partial_\mu U)^\dagger (\partial^\mu U). \quad (5)$$

(b) Using (4), express $\Delta\mathcal{L}$ from S and $\vec{\xi}$ retaining only the terms up to $\mathcal{O}(1/v)$.

Hint. You might find the following relation useful: $\partial_\mu \xi = (\vec{\xi} \cdot \partial_\mu \vec{\xi})/\xi$.

(c) What kind of vertices do these term represent?