Due by: 14 March

Exercise 1. ***Pion decay to leptons***

Consider the Lagrangian for semileptonic weak interactions:

$$\mathcal{L} = \frac{4G_F}{\sqrt{2}} \left(\bar{\ell}_L \gamma^\mu \nu_L\right) \left(\bar{u}_L \gamma_\mu d_L\right) + \text{h.c.} \tag{1}$$

with $\nu_L = P_L \nu = \frac{1}{2}(1-\gamma^5)\nu$.

(a) Using the quark currents defined as

$$J^{\mu a} = \bar{Q}\gamma^\mu \tau^a Q \qquad J^{\mu 5a} = \bar{Q}\gamma^\mu\gamma^5\tau^a Q \tag{2}$$

where $Q = \begin{pmatrix} u \\ d \end{pmatrix}$ is the quark doublet and $\tau^a = \sigma^a/2$ are the generators of $SU(2)$, show that

$$\bar{u}_L \gamma^\mu d_L = \frac{1}{2}\left(J^{\mu 1} + iJ^{\mu 2} - J^{\mu 51} - iJ^{\mu 52}\right). \tag{3}$$

The matrix element of $J^{\mu 5a}$ between the vacuum and an on-shell pion can be written as

$$\langle 0|J^{\mu 5a}|\pi^b(p)\rangle = -ip^\mu f_\pi \delta^{ab} e^{-ip\cdot x} \tag{4}$$

where f_π is a constant with the dimension of a mass.

(b) Using this identification together with the result of part (a), show that the amplitude for the decay $\pi^+ \to \ell^+\nu$ ($|\pi^+\rangle = \frac{1}{\sqrt{2}}(|\pi^1\rangle + i\,|\pi^2\rangle)$) is given by

$$i\mathcal{M} = G_F f_\pi \bar{u}(q)\not{p}(1-\gamma^5)v(k) \tag{5}$$

where p, k and q are the momenta of the π^+, ℓ^+ and ν respectively.

The decay rate of a particle of mass m through the channel corresponding to $\mathcal{M}$ is given by

$$\Gamma = \frac{1}{2m}\int d\Phi_{1\to 2}\overline{|\mathcal{M}|^2}, \tag{6}$$

where the phase-space measure is

$$d\Phi_{1\to 2} = \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}(2\pi)\delta(p_1^2-m_1^2)(2\pi)\delta(p_2^2-m_2^2)(2\pi)^4\delta^{(4)}(p-p_1-p_2), \tag{7}$$

and $\overline{|\mathcal{M}|^2}$ is the spin-summed and averaged squared matrix element.

(c) Compute the (partial) decay rate of the pion for this channel.

(d) Show that it vanishes in the limit of zero lepton mass.

(e) Show that the relative rate of pion decay to muons and electrons is given by

$$\frac{\Gamma(\pi^+ \to e^+\nu)}{\Gamma(\pi^+ \to \mu^+\nu)} = \left(\frac{m_e}{m_\mu}\right)^2 \frac{\left(1 - m_e^2/m_\pi^2\right)^2}{\left(1 - m_\mu^2/m_\pi^2\right)^2} \approx 10^{-4}. \tag{8}$$

(f) From the measured pion lifetime, the Fermi constant G_F, and the pion and muon masses determine the value of f_π.

Hint. You might want to take a look at http://pdg.lbl.gov/. Remark that different sources use different definitions of f_π who differ by factors of $(\sqrt{2})^n$.

Exercise 2. ***Pion decay to photons***

You have seen in the lecture that the matrix element associated with the decay $\pi^0 \to \gamma\gamma$ is – taking into account the axial anomaly:

$$i\mathcal{M} = iA\epsilon_\nu^*\epsilon_\lambda^*\varepsilon^{\nu\lambda\alpha\beta}k_\alpha q_\beta, \qquad A = \frac{e^2}{4\pi^2}\frac{1}{f_\pi} \tag{9}$$

where k and q are the momentum of the two photons and their polarization vectors.

(a) Compute the (partial) decay rate of the pion for this channel.

Note that there is an extra factor $\frac{1}{2}$ coming from the symmetry of the final-state phase-space.