Integral reduction and five-loop supergravity

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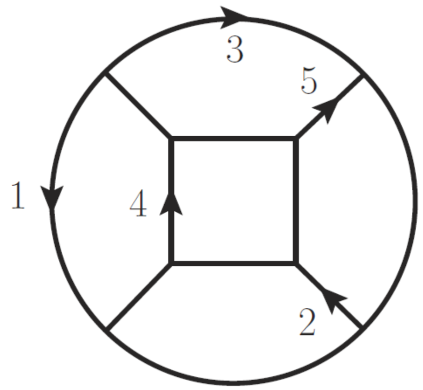
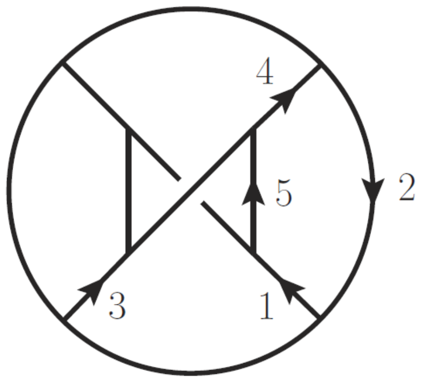
June 5 2018, Taming the Complexity of Multiloop Integrals, ETH Zurich





With *Zvi Bern, John Joseph Carrasco, Wei-Ming Chen, Alex Edision, Michael Enciso, Henrik Johansson, Julio Parra-Martinez, Radu Roiban*

arXiv:1703.08927, JHEP 1705 (2017) 137
arXiv:1708.06807, Phys. Rev. D. 96, 126012
arXiv:1804.09311, accepted by Phys. Rev. D.

Outline

- Background
- Results for UV properties of $\mathcal{N} = 8$ supergravity at 5 loops
- Maximal-cut reduction of UV divergences
- Full reduction from “UV-region IBP”
- All-loop patterns & future outlook

What's the UV behavior of $\mathcal{N} = 8$ SUGRA?

- Previous calculations: finite up to 4 loops, $D_c = 4 + 6/L$

  Bern, Carrasco, Dixon, Johansson, Kosower, Roiban 2007
  Bern, Carrasco, Dixon, Johansson, Roiban 2009, 2012

- Symmetry arguments: divergent at 7 loops, or 5 loops at $D_c = 24/5$, due to counterterm $\sim D^8 R^4$.

  Green, Russo, Vanhove 2010; Bossard, Howe, Stelle 2011; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger 2010; Vanhove 2010; Bjornsson, Green 2010; Bjornsson 2010

- Inadequacy of symmetry arguments (in other SUGRA theories)

  $N = 4$ *finite in* $D = 5$ *at 2 loops:* Bern, Davies, Dennen, Huang 2012
  $N = 4$ *finite in* $D = 4$ *at 3 loops:* Bern, Davies, Dennen, Huang 2012
  $N = 5$ *finite in* $D = 4$ *at 4 loops*: Bern, Davies, Dennen 2014

- This talk: the 5-loop calculation & paths to higher loops

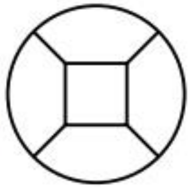
The five-loop results

[Bern, Carrasco, Chen, Edison, Johansson, Parra-Martinez, Roiban, MZ, 2018]

- $\mathcal{N}=8$ SUGRA is UV finite in $D=22/5$, confirming symmetry predictions.

- $\mathcal{N}=8$ SUGRA diverges in $D=24/5$, as positive-definite vacuum integrals. $\Longrightarrow$ Nonzero coefficient of D^8R^4.

$$\mathcal{M}_4^{(5)}\Big|_{\text{leading}} = -\frac{16\times 629}{25}\left(\frac{\kappa}{2}\right)^{12}(s^2+t^2+u^2)^2 stu M_4^{\text{tree}}\left(\frac{1}{48}\right.$$



$$+\frac{1}{16}$$

$$\left.\right)$$

$$= -17.9\left(\frac{\kappa}{2}\right)^{12}\frac{1}{(4\pi)^{12}}(s^2+t^2+u^2)^2 stu M_4^{\text{tree}}\,\frac{1}{\epsilon}\,.$$

[FIESTA: Smirnov, Smirnov, Tentyukov]

Challenges in a 5-loop calculation

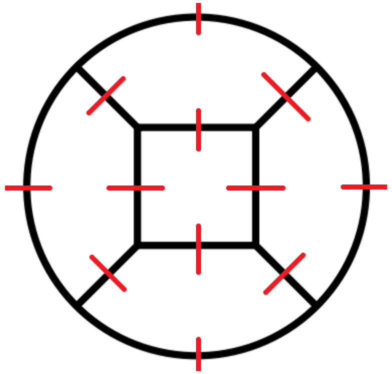
1) The loop integrand:
   Explosion of terms in a Feynman diagram approach
   Solutions: *(Generalized) double copy, unitarity cuts*

$$\mathcal{N}_{GR} \sim \mathcal{N}_{YM} \tilde{\mathcal{N}}_{YM} + J\tilde{J}$$

$$J \sim \text{BCJ discrepancy function}$$

[Bern, Carrasco, Chen, Johansson, Roiban 2017]

2) Integration in UV region:
   Large number of vacuum integrals, high-degree numerators
   Solutions: *finding better integrand, Unitarity cuts of vacuum integrals*



= ?

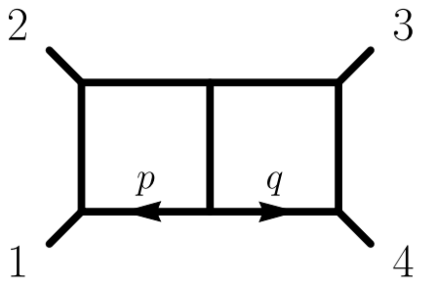
Warmup: 2-loop $\mathcal{N}=4$ SUGRA in $D=5$

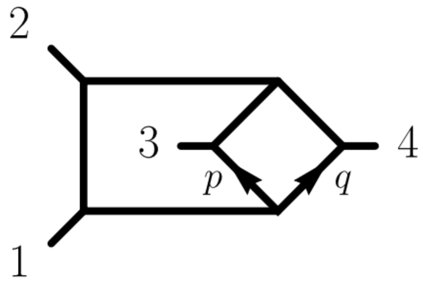
[Bern, Enciso, Parra-Martinez, MZ, 2018]

$$\mathcal{N}=4\ \text{SUGRA} \cong (\mathcal{N}=4)\otimes(\mathcal{N}=0)$$

$$\mathcal{A}_{\text{SUGRA}} = \sum_i \frac{n_i^{\text{SYM}}\cdot n_i^{\text{YM}}}{\text{propagators}}$$

Enhanced cancellation from double copy
[Bern, Davies, Dennen, Huang 2012]



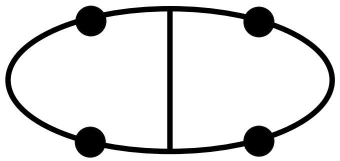


(++++) numerators (both diagrams)

$$\lambda_p^2\lambda_q^2 + \lambda_p^2\lambda_{p+q}^2 + \lambda_q^2\lambda_{p+q}^2$$

1st step: vacuum expansion

$$\int d^5p\, d^5q\, \frac{\lambda_p^2\lambda_q^2 + \lambda_p^2\lambda_{p+q}^2 + \lambda_q^2\lambda_{p+q}^2}{(p^2)^A(q^2)^B[(p+q)^2]^C}$$

 $\equiv I_{3,1,3}$

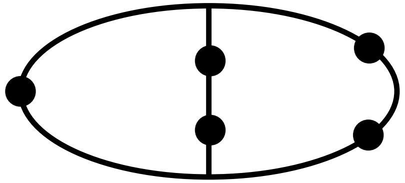
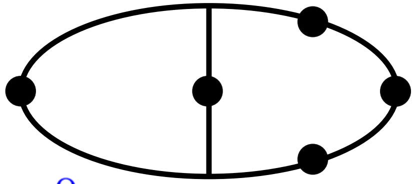
2nd step: Lorentz invariance

$$\frac{3}{70}\int d^5p\, d^5q\, \frac{p^2+q^2+(p+q)^2}{(p^2)^A(q^2)^B[(p+q)^2]^C}$$

(Planar) + (Nonplanar) $\propto I_{1,2,2} + 2I_{1,1,3} \sim 0\,?$

3rd step: integration …

Maximal-cut vacuum integrals

 $+\ 2$  $= I_{1,2,2} + 2I_{1,1,3} = \text{UV finite?}$

$= -\dfrac{\pi}{192\epsilon} + \mathcal{O}(\epsilon^0)$ $\qquad$ $= \dfrac{\pi}{96\epsilon} + \mathcal{O}(\epsilon^0)$

No one-loop divergence in 5D, UV from max. cut! $\dfrac{1}{l^2 - m^2} \to \delta(l^2 - m^2)$

Classic use of cuts: *cut integrand* = product of trees

Also consider *cut integrals*, defined on contours preserving integral relations

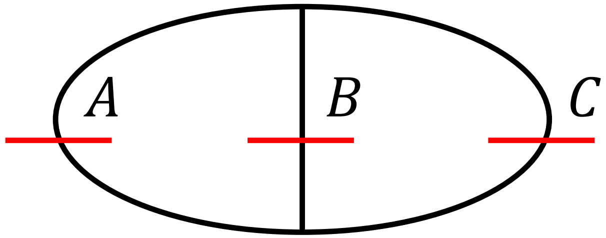
Kosower, Larsen, 2012; Caron-Huot, Larsen, 2012; Sogaard, 2013; Johansson, Kosower, Larsen, 2013; Sogaard, Zhang, 2013; Sogaard, Zhang, 2014; Primo, Tancredi, 2016, 2017; Abreu, Britto, Durh, Gardi, 2017; Bosma, Sogaard, Zhang 2017

Maximal-cut vacuum integrals

Baikov representation of Feynman integrals [Baikov, 1996]

$$I_{A,B,C} = \int d^5p\, d^5q \, \frac{1}{(p^2)^A (q^2)^B [(p+q)^2]^C}$$

$$\propto \int \frac{dz_1}{z_1^A} \frac{dz_2}{z_2^B} \frac{dz_3}{z_3^C} P(z_1, z_2, z_3),$$



$$\text{where } P(z_1, z_2, z_3) = 2z_1 z_2 + 2z_2 z_3 + 2z_3 z_1 - z_1^2 - z_2^2 - z_3^2$$

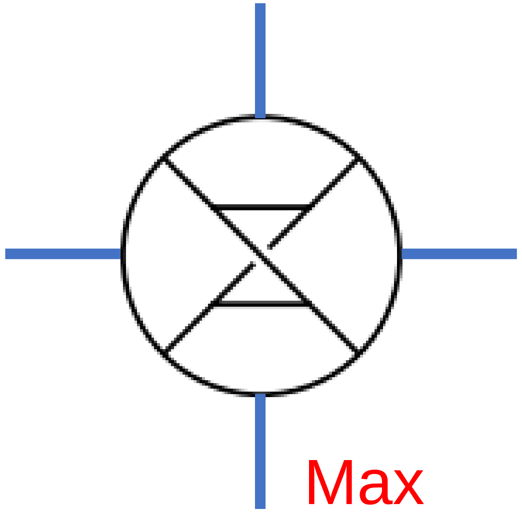
Cuts from contour prescription [Sogaard, Zhang, 2014]

$$\int \frac{dz}{z^A} \to \oint_{\Gamma_\epsilon(0)} \frac{dz}{z^A}. \qquad I_{A,B,C} = \text{Coeff of } z_1^{A-1} z_2^{B-1} z_3^{C-1} \text{ in } P(z_1, z_2, z_3)$$

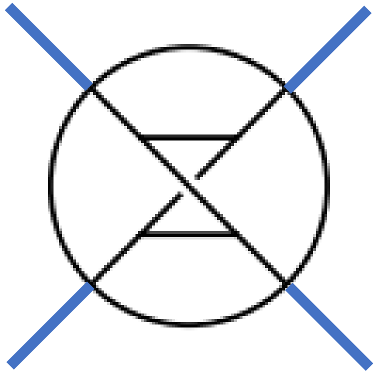
Indeed $I_{1,2,2} + 2I_{1,1,3} \sim 0\,.$ ✓ UV finite.

Max. cut calculation at 5 loops

1) Vacuum expansion, only from diagrams containing top-level vacuums



Max



up to N^4Max

2) Apply Lorentz invariance See also: Mastrolia, Peraro, Primo, 2016

3) Integration of vacuums on max. cut

Max. cut calculation at 5 loops

- Analytic integration possible in some cases, e.g. crossed cube topology
  [Bern, Carrasco, Chen, Johansson, Roiban, MZ, 2017]
- Max. cut Baikov polynomial $z_1 = z_2 = \cdots = z_{12} = 0,$

$$P(z_i)\big|_{\text{cut}} = z_{13} z_{14} z_{15} (z_{13} - z_{14})(z_{13} + z_{14} + z_{15})$$

$$\text{cut integral} = \int dz_{13} \int dz_{14} \int dz_{15}\, z_{13}^{y_1}\, z_{14}^{y_2}\, z_{15}^{y_3}\, P(z_i)\big|_{\text{cut}}^{-(3+y_1+y_2+y_3)/5}$$

- Integration contour with boundary $P(z_i)\big|_{\text{cut}} = 0\,.$
  [Bosma, Sogaard, Zhang, 2017]

Max. cut calculation at 5 loops

- Analytic integration possible in some cases, e.g. crossed cube topology
  [Bern, Carrasco, Chen, Johansson, Roiban, MZ, 2017]
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$$= \frac{\Gamma((1+2y_1-3y_2+2y_3)/5)}{\Gamma((3+y_1+y_2+y_3)/5)\Gamma((4+3y_1-2y_2+3y_3)/5)}$$

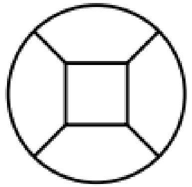
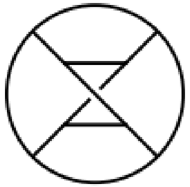
$$\times\, \Gamma((2+4y_1-y_2-y_3)/5)\, \Gamma((2-y_1+4y_2-y_3)/5)$$

$$\times\, \Gamma((2-y_1-y_2+4y_3)/5)$$

Max. cut calculation at 5 loops

- Generic cases: *unitarity-compatible integration-by-parts (IBP) reduction*.
  [Gluza, Kajda, Kosower, 2010; Ita, 2015; Larsen, Zhang, 2015...] Talks by Harald Ita, Kasper Larsen
- Dimension shifting to remove dots; syzygy to control propagator power.
- Linear relations between max.-cut vacuum integrals. Solve small linear system of size ~ 500.

$$D=\frac{22}{5}:\ \mathcal{M}_4^{(5)}\Big|_{\text{leading}} \propto\ 0$$

$$D=\frac{24}{5}:\ \mathcal{M}_4^{(5)}\Big|_{\text{leading}} \propto\ \left(\frac{1}{48}\right.$$

$$+\frac{1}{16}$$

$$\left.\right)$$

Surprisingly, both are the full results!

Aside: dimension shifting

[Bern, Dixon, Kosower, 1992; Tarasov, 1996]

$$\mathcal{I}_1 = \int \frac{dz_1}{z_1} \ldots \int \frac{dz_n}{z_n} dz_{n+1} \ldots dz_{n+m} \mathcal{N}(z_i) F(z_i)^{\frac{D-E-L-1}{2}}$$

$$= \int \frac{dz_1}{z_1} \ldots \int \frac{dz_n}{z_n} dz_{n+1} \ldots dz_{n+m} \left[\mathcal{N}(z_i) F(z_i)\right] F(z_i)^{\frac{(D-2)-E-L-1}{2}}$$

$$\mathcal{I}_2 = \int \frac{dz_1}{z_1^2} \ldots \int \frac{dz_n}{z_n} dz_{n+1} \ldots dz_{n+m} \mathcal{N}(z_i) F(z_i)^{\frac{D-E-L-1}{2}}$$

$$= \int \frac{dz_1}{z_1} \ldots \int \frac{dz_n}{z_n} dz_{n+1} \ldots dz_{n+m} \frac{\partial}{\partial z_1} \left[\mathcal{N}(z_i) F(z_i)^{\frac{D-E-L-1}{2}}\right]$$

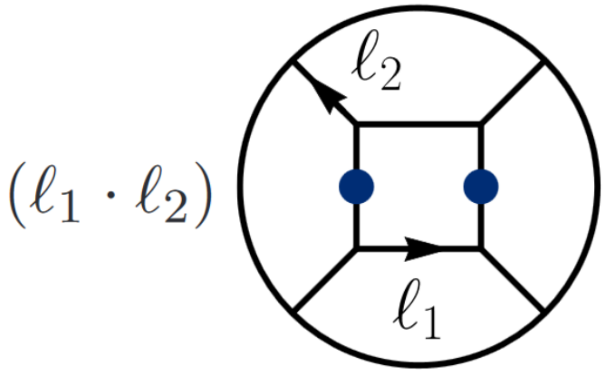
[Zhang, 2016]

Formulas simplified on maximal cut.

Full calculation at 5 loops

- Want full results without assumptions.

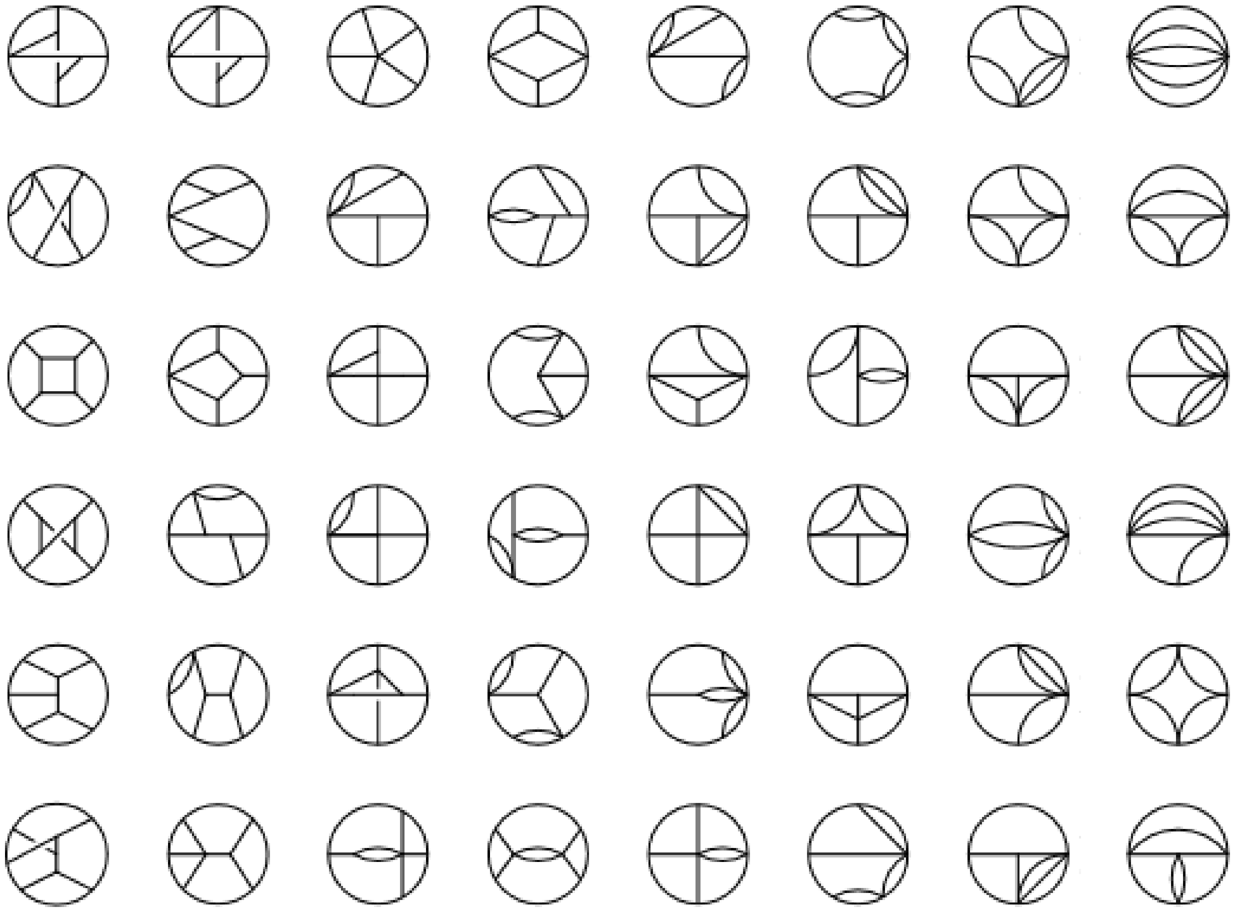
- Old integrand: spurious quartic divergence in $D = 24/5$ (contacts worse). ~ 17 million distinct vacuum integrals (up to 6 dots).



- Constructed improved integrand: top level diagrams log. divergent ~ 140 thousand distinct vacuum integrals (up to 4 dots).

Five-loop vacuum topologies

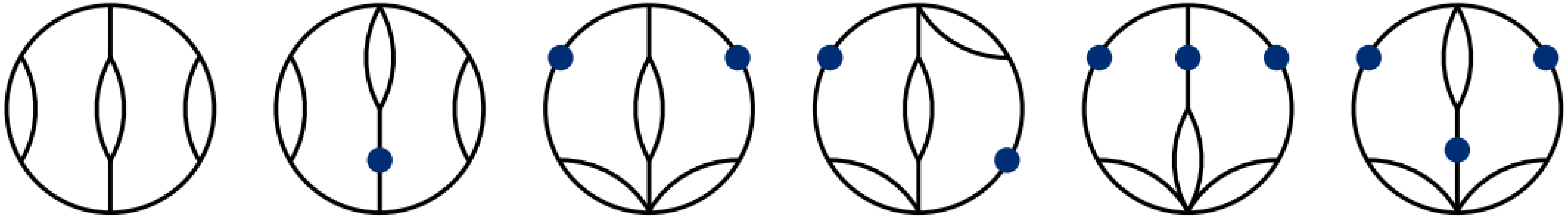
[Thomas Luthe 2015 thesis]



More propagators ⟷ **Fewer propagators**

Integral identification

- Finding identical integrals: graph isomorphism + extra enhancement: bubble ↔ propagator

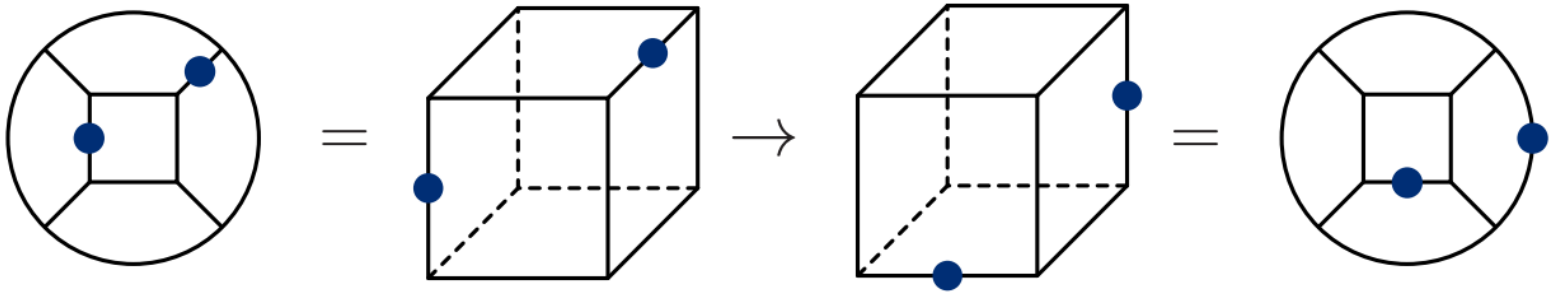


- Alternative approach: canonical form of $\mathcal{U}, \mathcal{F}$ polynomials. Systematic but slower, used to check results.

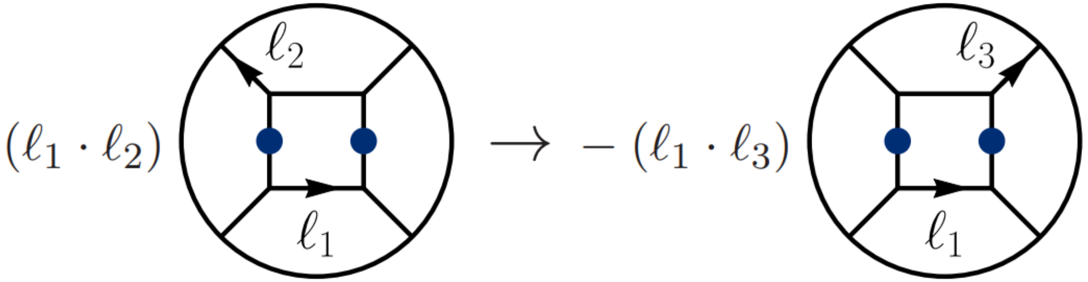
[A. Pak, 2011; J. Hoff, 2016]

Graph symmetry relations

- Dots moved to “canonical” positions.



- “Dot-preserving” symmetry gives more integral relations.
  Combined with IBP relations in linear system.



IBP from SL(L) relabeling symmetry

- Feynman integrals have linear relations from *integration by parts.*

  [Chetyrkin, Tkachev, 1981]

- Usual IBP contains redundant information (details of IR regularization); only need relations *between UV poles*

  [Bern, Carrasco, Dixon, Johansson, Kosower, Roiban 2007
  Bern, Carrasco, Dixon, Johansson, Roiban 2009, 2012]

- Systematic construction: [Bern, Enciso, Parra-Martinez, MZ, 2018]

  Infinitesimal SL(L) relabeling symmetry $\Delta l_i^\mu = \omega_{ij} l_j^\mu$

  acting on all log. divergent vacuums

Two-loop example for UV IBP

$$0 = \int d^D l_1 \int d^D l_2 \left(l_1^\mu \frac{\partial}{\partial l_1^\mu} - l_2^\mu \frac{\partial}{\partial l_2^\mu} \right) \frac{1}{(l_1^2 - m^2)^A (l_2^2 - m^2)^B \left[(l_1 + l_2)^2 - m^2 \right]^C}$$

$$= (-2A + 2B) I_{A,B,C} - 2C\, I_{A-1,B,C+1} + 2C I_{A,B-1,C+1}$$

$$+ m^2 \left(-2A\, I_{A+1,B,C} + 2B\, I_{A,B+1,C} \right) .$$

UV finite

with $A + B + C = 5$. Setting $A = 1,\ B = C = 2$,

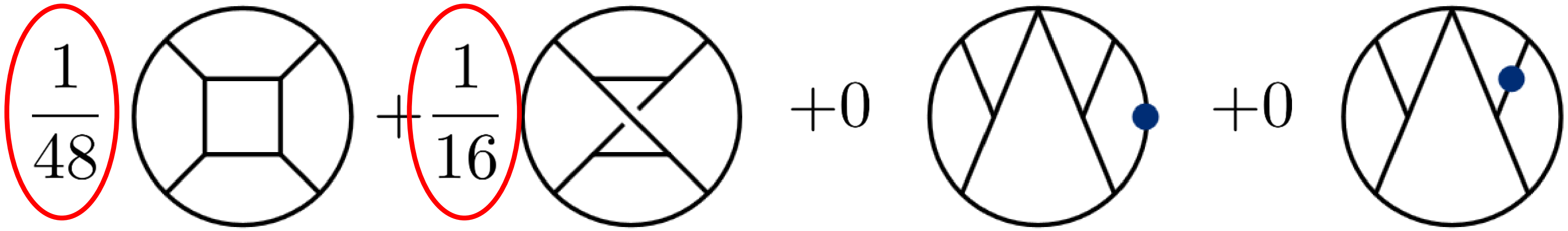
$$2I_{1,2,2} - 4I_{0,2,3} + 4I_{1,1,3} = \text{UV finite}$$

General prescription: (1) traceless SL(L) generators; (2) Set $m = 0$; (3) Drop factorized topologies. (4) optional: set $\epsilon = 0$ to kill trace term

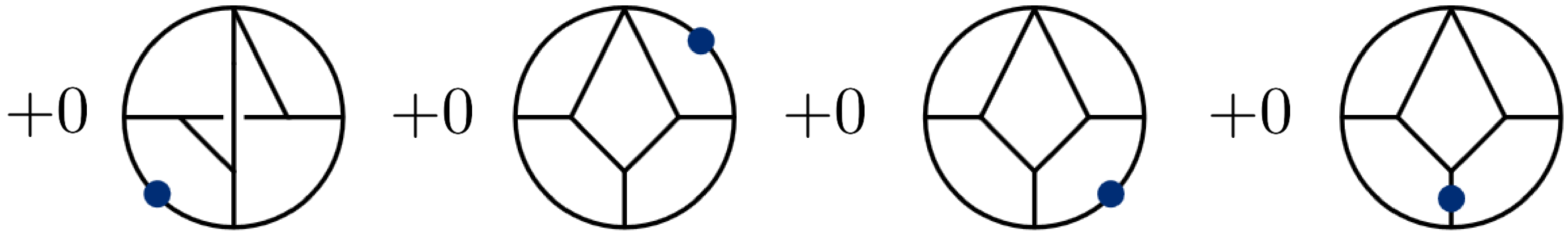
Full calculation results

- In $D = 22/5$: UV finite, as expected.
- In $D = 24/5$: 2.8 million relations between 0.85 million integrals, ~1 billion nonzero entries. 8 master integrals. Sparse Gaussian elimination over finite fields.

  [Schabinger, von Manteuffel, 2014; Peraro, 2016]
- Summing up diagrams… Cancellation of lower-level coefficients! No triangles?

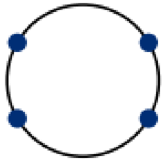


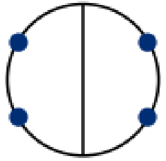
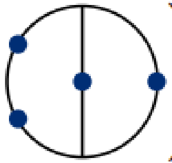
=diagram symmetry factors! $|S_4 \times S_2|$, $|D_8|$

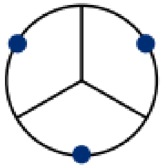
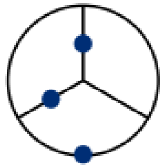


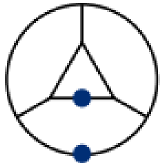
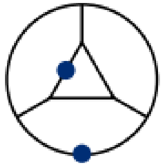
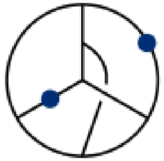
All-loop patterns

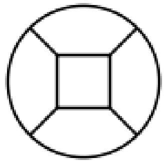
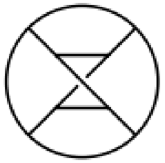
Cross-order relations from removing propagators

$$\mathcal{M}_4^{(1)}\Big|_{\text{leading}} = -3\,\mathcal{K}_{\rm G}\left(\frac{\kappa}{2}\right)^4$$  ,

$$\mathcal{M}_4^{(2)}\Big|_{\text{leading}} = -8\,\mathcal{K}_{\rm G}\left(\frac{\kappa}{2}\right)^6 (s^2+t^2+u^2)\left(\frac{1}{4}\right.$$  $$+\frac{1}{4}$$  $$\left.\right),$$

$$\mathcal{M}_4^{(3)}\Big|_{\text{leading}} = -60\,\mathcal{K}_{\rm G}\left(\frac{\kappa}{2}\right)^8 stu\left(\frac{1}{6}\right.$$  $$+\frac{1}{2}$$  $$\left.\right),$$

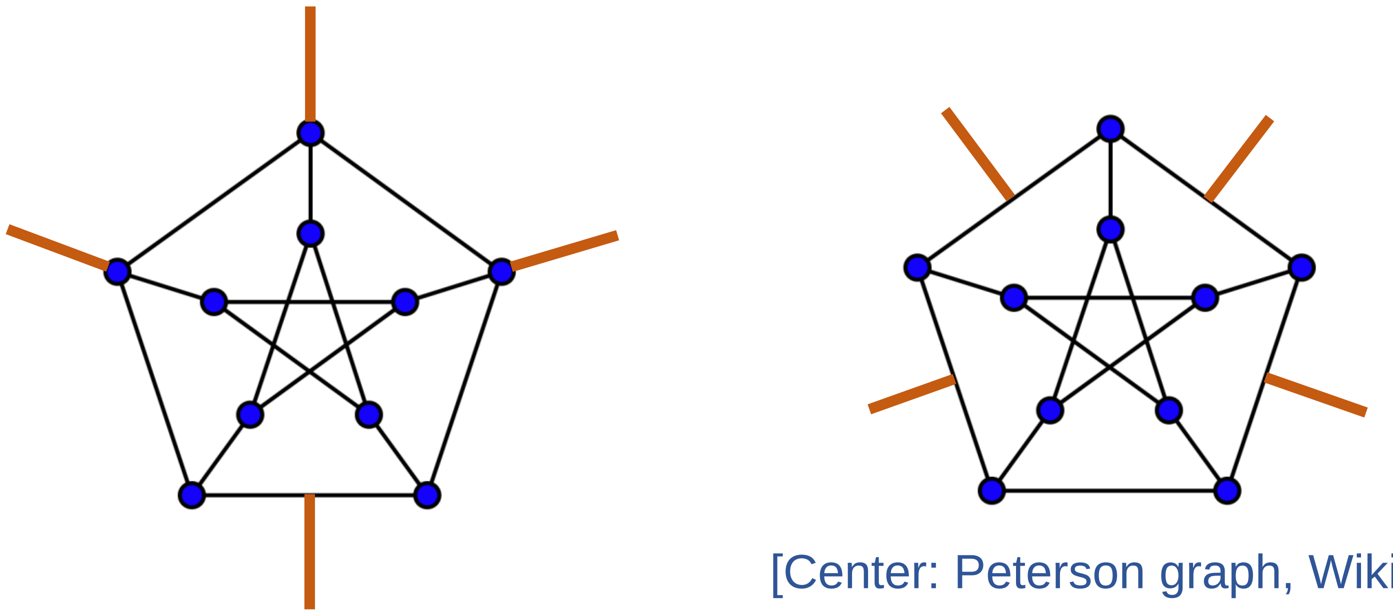
$$\mathcal{M}_4^{(4)}\Big|_{\text{leading}} = -\frac{23}{2}\,\mathcal{K}_{\rm G}\left(\frac{\kappa}{2}\right)^{10} (s^2+t^2+u^2)^2\left(\frac{1}{4}\right.$$  $$+\frac{1}{2}$$  $$+\frac{1}{4}$$  $$\left.\right),$$

$$\mathcal{M}_4^{(5)}\Big|_{\text{leading}} = -\frac{16\times 629}{25}\,\mathcal{K}_{\rm G}\left(\frac{\kappa}{2}\right)^{12} (s^2+t^2+u^2)^2\left(\frac{1}{48}\right.$$  $$+\frac{1}{16}$$  $$\left.\right),$$

No triangles + BCJ-like symmetry factors

Outlook – higher loops

- Gauge invariant sub-component of UV divergence (top-level master coefficients) may be extracted from a subset of 4-point diagrams.



[Center: Peterson graph, Wikipedia]

- Max. cut simplifies vacuum integral reduction.
- Conjectural all-loop patterns constrain results.

Conclusions

- Extracted UV properties of $\mathcal{N} = 8$ supergravity at 5 loops. Implication for 4D unclear.

- Integral reduction, and interplay with unitarity, plays an important role. Various ideas to control size of linear system.

- Cuts of vacuum integrals dramatically simplify calculation, pointing to paths to higher loops.

- Surprising patterns deserving further understanding.

Thank you!