Integration-by-parts reductions via unitarity cuts and algebraic geometry

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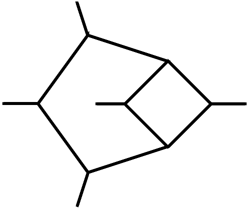


Taming the complexity of multi-loop integrals

ETH Zurich, 4th of June 2018

Based on PRD **93**(2016)041701, 1712.09737 and 1805.01873
with Yang Zhang et al.

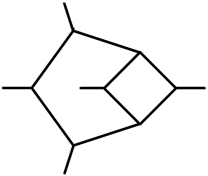
Overview

1. Motivation

2. Baikov representation

3. IBP identities on cuts

4. Syzygy equation and its solution

5. Main example: 

Integration-by-parts reductions

Integration-by-parts identities arise from the vanishing integration of total derivatives,
[Chertyrkin, Tchakov, Nucl. Phys. B **192**, 159 (1981)]

$$\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\pi^{D/2}} \sum_{j=1}^{L} \frac{\partial}{\partial \ell_j^\mu} \frac{v_j^\mu \, P}{D_1^{a_1} \cdots D_k^{a_k}} \; = \; 0$$

where P and v_j^μ are polynomials in ℓ_i, p_j, and $a_i \in \mathbb{N}$.

Role in perturbative QFT calculations:

- **Reduction.** IBP identities reduce any set of loop integrals to a *typically much smaller set* of master integrals.
- **Computing master integrals.** Using IBP reduction, the master integrals $\mathcal{I}_j$ can be computed via differential equations:
[T. Gehrmann and E. Remiddi, Nucl. Phys. B **580**, 485 (2000)]

$$\frac{\partial}{\partial x_m} \boldsymbol{\mathcal{I}}(\boldsymbol{x}, \epsilon) = A_m(\boldsymbol{x}, \epsilon) \boldsymbol{\mathcal{I}}(\boldsymbol{x}, \epsilon)$$

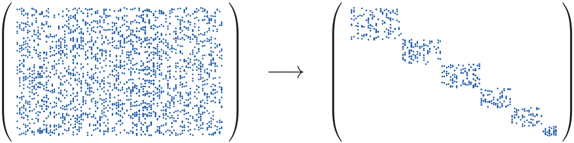
where x_m denotes a kinematical invariant.

IBP reductions on unitarity cuts

Standard approach: enumerate all linear relations and apply Gauss-Jordan elimination to *large* linear systems

[Laporta, Int.J.Mod.Phys. A **15** (2000) 5087-5159]

Idea here: use *unitarity cuts* to block-diagonalize system



We use the Baikov representation $\left(k = \frac{L(L+1)}{2} + LE\right)$,

$$I(N; a) \equiv \int \prod_{j=1}^{L} \frac{\mathrm{d}^D \ell_j}{i\pi^{D/2}} \frac{N}{D_1^{a_1} \cdots D_k^{a_k}} = \int \frac{\mathrm{d}z_1 \cdots \mathrm{d}z_k}{z_1^{a_1} \cdots z_k^{a_k}} \underset{(\widehat{\boldsymbol{p}},\boldsymbol{\ell})}{\mathrm{Gram}}(z)^{\frac{D-L-E-1}{2}} N$$

[Baikov, Phys.Lett. B **385** (1996) 404-410]

in which cuts are straightforward to apply,

$$\int \frac{\mathrm{d}z_i}{z_i^{a_i}} \xrightarrow{\text{cut}} \oint_{\Gamma_\epsilon(0)} \frac{\mathrm{d}z_i}{z_i^{a_i}} \qquad i \in \mathcal{S}_{\text{cut}}$$

Consider a generic loop integral,

$$I(N;\alpha_1,\ldots,\alpha_m;D) = \int \prod_{j=1}^{L} \frac{\mathrm{d}^D \ell_j}{i\pi^{D/2}} \frac{\overbrace{D_{k+1}^{\alpha_{k+1}} \cdots D_m^{\alpha_m}}^{N}}{D_1^{\alpha_1} \cdots D_k^{\alpha_k}}$$

Let $\{v_1,\ldots,v_{E+L}\} \equiv \{p_1,\ldots,p_E,\ell_1,\ldots,\ell_L\}$ and, with $x_{i,j} = v_i \cdot v_j$

$$U = \begin{vmatrix} x_{1,1} & \cdots & x_{1,E} \\ \vdots & \ddots & \vdots \\ x_{E,1} & \cdots & x_{E,E} \end{vmatrix} \quad \text{and} \quad F = \begin{vmatrix} x_{1,1} & \cdots & x_{1,E+L} \\ \vdots & \ddots & \vdots \\ x_{E+L,1} & \cdots & x_{E+L,E+L} \end{vmatrix}.$$

The Baikov variables are the inverse propagators $\left(m = LE + \frac{L(L+1)}{2}\right)$

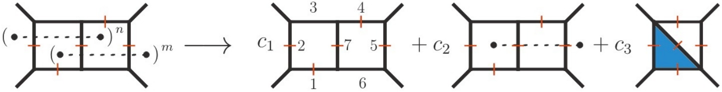
$$z_\alpha = D_\alpha = \sum_{\beta=1}^{m} A_{\alpha\beta} x_\beta + \sum_{1\le i\le j\le E} (B_\alpha)_{ij}\lambda_{ij} \quad \text{with} \quad A_{\alpha\beta}, (B_\alpha)_{ij} \in \mathbb{Z}$$

The Baikov representation is

$$I(N;\alpha;D) \propto U^{\frac{E-D+1}{2}} \int \mathrm{d}z_1 \cdots \mathrm{d}z_m \frac{z_{k+1}^{\alpha_{k+1}} \cdots z_m^{\alpha_m}}{z_1^{\alpha_1} \cdots z_k^{\alpha_k}} F^{\frac{D-L-E-1}{2}}$$

Example: Zurich-flag cut

Let us find the IBP reductions of the double-box integral. We start by allowing *only integrals which contain all Zurich-flag propagators:*



Define $S_{\text{cut}} = \{1, 2, 4, 5, 7\}$.

After cutting $\frac{1}{\tilde{z}_i} \to \delta(\tilde{z}_i)$, $i \in S_{\text{cut}}$, the double-box integral takes the form

$$I^{\text{DB}}_{\text{cut}}[P] \;=\; \int \prod_{i=1}^{9} \mathrm{d}\tilde{z}_i \frac{F(\tilde{\boldsymbol{z}})^{\frac{D-6}{2}}}{\tilde{z}_3\,\tilde{z}_6} \prod_{j \in S_{\text{cut}}} \delta(\tilde{z}_j)\, P(\tilde{\boldsymbol{z}})$$

As the cut sets $\tilde{z}_{\{1,2,4,5,7\}}$ to zero, we set $z_{\{1,2,3,4\}} = \tilde{z}_{\{3,6,8,9\}}$ in the following.

Generic total derivative

After integrating out the delta functions and relabeling we have

$$I_{\text{cut}}^{\text{DB}}[P] = \int \frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4}{z_1 z_2} F(\boldsymbol{z})^{\frac{D-6}{2}} P(\boldsymbol{z})\,.$$

An IBP relation corresponds to a total derivative or, equivalently, an exact diff. form. The generic exact diff. form of the form $I_{\text{cut}}^{\text{DB}}$ is

$$\begin{aligned}
0 &= \int \mathrm{d}\left[\sum_{i=1}^{4} \frac{(-1)^{i+1} a_i(\boldsymbol{z}) F(\boldsymbol{z})^{\frac{D-6}{2}}}{z_1 z_2} \mathrm{d}z_1 \wedge \cdots \wedge \widehat{\mathrm{d}z_i} \wedge \cdots \wedge \mathrm{d}z_4\right] \\
&= \int \left[\sum_{i=1}^{4} \frac{\partial}{\partial z_i}\left(\frac{a_i(\boldsymbol{z}) F(\boldsymbol{z})^{\frac{D-6}{2}}}{z_1 z_2}\right)\right] \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_4 \\
&= \int \left[\sum_{i=1}^{4}\left(\frac{\partial a_i}{\partial z_i} + \frac{D-6}{2F} a_i \frac{\partial F}{\partial z_i}\right) - \sum_{j=1,2} \frac{a_j}{z_j}\right] \frac{F(\boldsymbol{z})^{\frac{D-6}{2}}}{z_1 z_2} \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_4\,.
\end{aligned}$$

The red term corresponds to an integral in $(D-2)$ dimensions, and the purple term in general produces squared propagators.

IBP identities from syzygy equations

To get the generic exact form

$$0 = \int \left[\sum_{i=1}^{4} \left(\frac{\partial a_i}{\partial z_i} + \frac{D-6}{2F} a_i \frac{\partial F}{\partial z_i} \right) - \sum_{j=1,2} \frac{a_j}{z_j} \right] \frac{F(\boldsymbol{z})^{\frac{D-6}{2}}}{z_1 z_2} \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_4$$

to correspond to an IBP relation in D dimensions with only single-power propagators, we demand that each term is *polynomial*,

$$\sum_{i=1}^{4} \frac{D-6}{2F} a_i \frac{\partial F}{\partial z_i} = \widetilde{b} \quad \Longrightarrow \quad \sum_{i=1}^{4} a_i \frac{\partial F}{\partial z_i} + bF = 0 \ \ \left(\text{with} \ \ b = \tfrac{2}{6-D} \widetilde{b}\right)$$

$$a_j = \widetilde{b}_j z_j \quad \Longrightarrow \quad a_j + b_j z_j = 0 \ \ \left(\text{with} \quad b_j = -\widetilde{b}_j\right),$$

with a_i, b_i, b polynomials in z. Such equations, with polynomial solutions, are known in algebraic geometry as *syzygy equations*.

[Gluza, Kajda, Kosower, PRD**83**(2011)045012], [Schabinger, JHEP**01**(2012)077], [Ita, PRD**94**(2016)116015]

Obtain IBPs by plugging (a_i, b) into the top equation.
Note: (qa_i, qb) is also a solution, for polynomial q.

Strategy to solve syzygy equations

Solve syzygy equations with c cuts

$$\sum_{j=1}^{m-c} a_j \frac{\partial F}{\partial z_k} + bF = 0 \tag{1}$$

$$a_j + b_j z_j = 0\,, \quad j = 1, \ldots, k-c \tag{2}$$

as follows

1) Find syzygy generators $\mathcal{M}_1 = \Big\langle (a_1, \ldots, a_m, b),\ \ldots \Big\rangle$ of eq. (1) for the *off-shell case* $c = 0$.

2) The generators of eq. (2) are trivial:

$$\mathcal{M}_2 = \langle z_1 \mathbf{e}_1, \ldots, z_k \mathbf{e}_k, \mathbf{e}_{k+1}, \ldots, \mathbf{e}_m \rangle$$

3) Take module intersection $\mathcal{M}_1\big|_{\text{cut}} \cap \mathcal{M}_2\big|_{\text{cut}}$.

Syzygies from Laplace expansion

A generating set of solutions of

$$\sum_{\alpha=1}^{m} a_\alpha \frac{\partial F}{\partial z_\alpha} \; + \; bF = 0$$

can be obtained from Gröbner basis calculations (Schreyer's thm).

also: [Bern, Enciso, Ita, Zeng, PRD **96**(2017)096017]

F is a determinant $\longrightarrow$ **solutions can be explicitly found!**

Laplace expansion of generic matrix:

$$\left[\sum_{k=1}^{n} r_{jk} \frac{\partial(\det R)}{\partial r_{ik}}\right] - \delta_{ij} \det R = 0$$

Syzygies from Laplace expansion

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also: [Bern, Enciso, Ita, Zeng, PRD **96**(2017)096017]

F is a determinant $\longrightarrow$ **solutions can be explicitly found!**

Laplace expansion of a symmetric matrix S:

$$\left[\sum_{k=1}^{n}(1+\delta_{ik})s_{jk}\frac{\partial(\det S)}{\partial s_{ik}}\right] - 2\delta_{ij}\det S = 0$$

Syzygies from Laplace expansion

A generating set of solutions of

$$\sum_{\alpha=1}^{m} a_\alpha \frac{\partial F}{\partial z_\alpha} + bF = 0$$

can be obtained from Gröbner basis calculations (Schreyer's thm).

also: [Bern, Enciso, Ita, Zeng, PRD **96**(2017)096017]

F is a determinant $\longrightarrow$ **solutions can be explicitly found!**

Laplace expansion of $S = \mathrm{Gram}(v_1, \ldots, v_{E+L})$:

$$\left[\sum_{k=1}^{E+L} (1+\delta_{ik}) x_{jk} \frac{\partial F}{\partial x_{ik}}\right] - 2\delta_{ij} F = 0$$

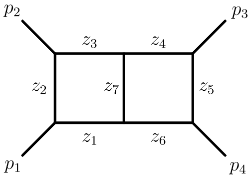
Using the chain rule this becomes

$$\sum_{\alpha=1}^{m} \left[\sum_{k=1}^{E+L} (1+\delta_{ik}) x_{jk} \frac{\partial z_\alpha}{\partial x_{ik}}\right] \frac{\partial F}{\partial z_\alpha} - 2\delta_{ij} F = 0 \qquad \begin{cases} E+1 \leq i \leq E+L \\ 1 \leq j \leq E+L \end{cases}$$

Proof (based on Józefiak complex) of completeness of syzygies.

[Böhm, Georgoudis, KJL, Schulze, Zhang, 1712.09737]

Example 1: syzygies of planar double box



Set $P_{12} = p_1 + p_2$ and

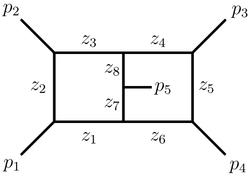
$$
\begin{aligned}
z_1 &= \ell_1^2\,, & z_2 &= (\ell_1 - p_1)^2\,, & z_3 &= (\ell_1 - P_{12})^2 \\
z_4 &= (\ell_2 + P_{12})^2\,, & z_5 &= (\ell_2 - p_4)^2\,, & z_6 &= \ell_2^2 \\
z_7 &= (\ell_1 + \ell_2)^2\,, & z_8 &= (\ell_1 + p_4)^2\,, & z_9 &= (\ell_2 + p_1)^2
\end{aligned}
$$

Only need to find explicit relation $z = Ax$. Here $A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ -2 & 0 & -2 & 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Set $t_{i,j} = (a_\alpha, b)$. The syzygy generators are *linear* in the z_k

$$
\begin{aligned}
t_{4,1} &= (z_1-z_2,\ z_1-z_2,\ -s+z_1-z_2,\ 0,\ 0,\ 0,\ z_1-z_2-z_6+z_9,\ t+z_1-z_2,\ 0,\ 0) \\
t_{4,2} &= (s+z_2-z_3,\ z_2-z_3,\ z_2-z_3,\ 0,\ 0,\ 0,\ z_2-z_3+z_4-z_9,\ -t+z_2-z_3,\ 0,\ 0) \\
t_{4,3} &= (-s+z_3-z_8,\ t+z_3-z_8,\ z_3-z_8,\ 0,\ 0,\ 0,\ z_3-z_4+z_5-z_8,\ z_3-z_8,\ 0,\ 0) \\
t_{4,4} &= (2z_1,\ z_1+z_2,\ -s+z_1+z_3,\ 0,\ 0,\ 0,\ z_1-z_6+z_7,\ z_1+z_8,\ 0,\ -2) \\
t_{4,5} &= (-z_1-z_6+z_7,\ -z_1+z_7-z_9,\ s-z_1-z_4+z_7,\ 0,\ 0,\ 0,\ -z_1+z_6+z_7,\ -z_1-z_5+z_7,\ 0,\ 0) \\
t_{5,1} &= (0,\ 0,\ 0,\ s-z_6+z_9,\ -t-z_6+z_9,\ z_9-z_6,\ z_1-z_2-z_6+z_9,\ 0,\ z_9-z_6,\ 0) \\
t_{5,2} &= (0,\ 0,\ 0,\ z_4-z_9,\ t+z_4-z_9,\ -s+z_4-z_9,\ z_2-z_3+z_4-z_9,\ 0,\ z_4-z_9,\ 0) \\
t_{5,3} &= (0,\ 0,\ 0,\ z_5-z_4,\ z_5-z_4,\ s-z_4+z_5,\ z_3-z_4+z_5-z_8,\ 0,\ -t-z_4+z_5,\ 0) \\
t_{5,4} &= (0,\ 0,\ 0,\ s-z_3-z_6+z_7,\ -z_6+z_7-z_8,\ -z_1-z_6+z_7,\ z_1-z_6+z_7,\ 0,\ -z_2-z_6+z_7,\ 0) \\
t_{5,5} &= (0,\ 0,\ 0,\ -s+z_4+z_6,\ z_5+z_6,\ 2z_6,\ -z_1+z_6+z_7,\ 0,\ z_6+z_9,\ -2)
\end{aligned}
$$

Example 2: syzygies of non-planar double pentagon



Set $P_{i,j} \equiv p_i + p_j$ and

$$z_1 = \ell_1^2\,, \qquad z_2 = (\ell_1 - p_1)^2\,, \qquad z_3 = (\ell_1 - P_{1,2})^2\,,$$
$$z_4 = (\ell_2 - P_{3,4})^2\,, \qquad z_5 = (\ell_2 - p_4)^2\,, \qquad z_6 = \ell_2^2\,,$$
$$z_7 = (\ell_1 + \ell_2)^2\,, \qquad z_8 = (\ell_1 + \ell_2 + p_5)^2\,, \qquad z_9 = (\ell_1 + p_3)^2\,,$$
$$z_{10} = (\ell_1 + p_4)^2\,, \qquad z_{11} = (\ell_2 + p_1)^2$$

Here $z = Ax$ with

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and the syzygy generators are again compact:

$$t_{5,1} = (z_1-z_2,\, z_1-z_2,\, -s_{1,2}+z_1-z_2,\, 0,\, 0,\, 0,\, z_1-z_2-z_6+z_{11},\, -s_{1,2}-s_{1,3}-s_{1,4}+z_1-z_2-z_6+z_{11},\, s_{1,3}+z_1-z_2,\, s_{1,4}+z_1-z_2,\, 0,\, 0)\,,$$

$$t_{5,2} = (s_{1,2}+z_2-z_3,\, z_2-z_3,\, z_2-z_3,\, 0,\, 0,\, 0,\, -s_{3,4}+z_1+z_2+z_4+z_7-z_8-z_9-z_{10}-z_{11},\, s_{1,3}+s_{1,4}+z_1+z_2+z_4+z_7-z_8-z_9-z_{10}-z_{11},\, s_{1,2}+s_{2,3}+z_2-z_3,\, -s_{1,3}-s_{1,4}-s_{2,3}-s_{3,4}+z_2-z_3,\, 0,\, 0)\,,$$

$$t_{5,3} = (z_9-z_1,\, -s_{1,3}-z_1+z_9,\, -s_{1,3}-s_{2,3}-z_1+z_9,\, 0,\, 0,\, 0,\, s_{3,4}-z_1-z_4+z_5+z_9,\, -s_{1,3}-s_{2,3}-z_1-z_4+z_5+z_9,\, z_9-z_1,\, s_{3,4}-z_1+z_9,\, 0,\, 0)\,,$$

$$t_{5,4} = (z_{10}-z_1,\, -s_{1,4}-z_1+z_{10},\, -s_{1,4}-s_{2,4}-z_1+z_{10},\, 0,\, 0,\, 0,\, -z_1-z_5+z_6+z_{10},\, s_{1,2}+s_{1,3}+s_{2,3}-z_1-z_5+z_6+z_{10},\, s_{3,4}-z_1+z_{10},\, z_{10}-z_1,\, 0,\, 0)\,,$$

$$t_{5,5} = (2z_1,\, z_1+z_2,\, -s_{1,2}+z_1+z_3,\, 0,\, 0,\, 0,\, z_1-z_6+z_7,\, -s_{1,2}+2z_1+z_3-z_6+z_7-z_9-z_{10},\, z_1+z_9,\, z_1+z_{10},\, 0,\, -2)\,,$$

$$t_{5,6} = (-z_1-z_6+z_7,\, -z_1+z_7-z_{11},\, s_{1,2}+s_{3,4}-2z_1-z_3-z_4+z_8+z_9+z_{10},\, 0,\, 0,\, 0,\, -z_1+z_6+z_7,\, s_{1,2}-2z_1-z_3+z_6+z_8+z_9+z_{10},\, s_{3,4}-z_1-z_4+z_5-z_6+z_7,\, -z_1-z_5+z_7,\, 0,\, 0)\,, \qquad (6.11)$$

$$t_{6,1} = (0,\, 0,\, 0,\, -s_{1,3}-s_{1,4}-z_6+z_{11},\, -s_{1,4}-z_6+z_{11},\, z_{11}-z_6,\, z_1-z_2-z_6+z_{11},\, -s_{1,2}-s_{1,3}-s_{1,4}+z_1-z_2-z_6+z_{11},\, 0,\, 0,\, z_{11}-z_6,\, 0)\,,$$

$$t_{6,2} = (0, 0, 0, s_{1,3}+s_{1,4}+z_1+z_3+z_4+z_7-z_8-z_9-z_{10}-z_{11},\, s_{1,3}+s_{1,4}+s_{2,3}+z_1+z_3+z_4+z_7-z_8-z_9-z_{10}-z_{11},\, -s_{1,2}-s_{3,4}+z_1+z_3+z_4+z_7-z_8-z_9-z_{10}-z_{11},\, -s_{3,4}+z_1+z_2+z_4+z_7-z_8-z_9-z_{10}-z_{11},\, s_{1,3}+s_{1,4}+z_1+z_2+z_4+z_7-z_8-z_9-z_{10}-z_{11},\, 0,\, 0,\, -s_{3,4}+z_1+z_3+z_4+z_7-z_8-z_9-z_{10}-z_{11},\, 0)\,,$$

$$t_{6,3} = (0,\, 0,\, 0,\, z_5-z_4,\, z_5-z_4,\, s_{3,4}-z_4+z_5,\, s_{3,4}-z_1-z_4+z_5+z_9,\, -s_{1,3}-s_{2,3}-z_1-z_4+z_5+z_9,\, 0,\, 0,\, s_{1,3}+s_{3,4}-z_4+z_5,\, 0)\,,$$

$$t_{6,4} = (0,\, 0,\, 0,\, -s_{3,4}-z_5+z_6,\, z_6-z_5,\, z_6-z_5,\, -z_1-z_5+z_6+z_{10},\, s_{1,2}+s_{1,3}+s_{2,3}-z_1-z_5+z_6+z_{10},\, 0,\, 0,\, s_{1,4}-z_5+z_6,\, 0)\,,$$

$$t_{6,5} = (0,\, 0,\, 0,\, z_1-z_6+z_7-z_9-z_{10},\, -z_6+z_7-z_{10},\, -z_1-z_6+z_7,\, z_1-z_6+z_7,\, -s_{1,2}+2z_1+z_3-z_6+z_7-z_9-z_{10},\, 0,\, 0,\, -z_2-z_6+z_7,\, 0)\,,$$

$$t_{6,6} = (0,\, 0,\, 0,\, -s_{3,4}+z_4+z_6,\, z_5+z_6,\, 2z_6,\, -z_1+z_6+z_7,\, s_{1,2}-2z_1-z_3+z_6+z_8+z_9+z_{10},\, 0,\, 0,\, z_6+z_{11},\, -2)\,.$$

Computing module intersections

Given $\mathcal{M}_1 = \langle v_1, \ldots, v_p \rangle$ and $\mathcal{M}_2 = \langle w_1, \ldots, w_q \rangle$ with v_i, v_w m-tuples of polynomials. Let Q denote the $m \times (p+q)$ matrix

$$Q = \begin{pmatrix} \vdots & & \vdots & \vdots & & \vdots \\ v_1 & \cdots & v_p & w_1 & \cdots & w_q \\ \vdots & & \vdots & \vdots & & \vdots \end{pmatrix}$$

Then compute wrt. POT and variable order $[z_1, \ldots, z_m] \succ [s_{ij}]$

$$\langle h_1, \ldots, h_t \rangle \equiv \text{Gröbner basis of column space of } \left(\begin{array}{ccc} & Q & \\ \hline 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{array} \right)$$

Selecting $h_i = (\overbrace{0, \ldots, 0}^{m}, x_1, \ldots, x_p, y_1, \ldots, y_q)$, we have

$$0 = \sum_{j=1}^{p} x_j v_j + \sum_{k=1}^{q} y_k w_k$$

Computing module intersections

Given $\mathcal{M}_1 = \langle v_1, \ldots, v_p \rangle$ and $\mathcal{M}_2 = \langle w_1, \ldots, w_q \rangle$ with v_i, v_w m-tuples of polynomials. Let Q denote the $m \times (p+q)$ matrix

$$Q = \begin{pmatrix} \vdots & & \vdots & \vdots & & \vdots \\ v_1 & \cdots & v_p & w_1 & \cdots & w_q \\ \vdots & & \vdots & \vdots & & \vdots \end{pmatrix}$$

Then compute wrt. POT and variable order $[z_1, \ldots, z_m] \succ [s_{ij}]$

$$\langle h_1, \ldots, h_t \rangle \equiv \text{Gröbner basis of column space of } \begin{pmatrix} & Q & \\ \hline 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

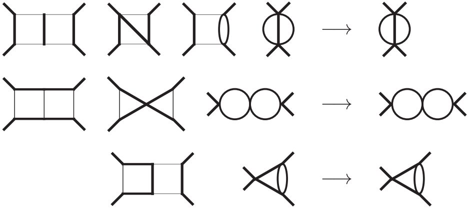
Selecting $h_i = (\overbrace{0, \ldots, 0}^{m}, x_1, \ldots, x_p, y_1, \ldots, y_q)$, we have

$$0 = \sum_{j=1}^{p} x_j v_j + \sum_{k=1}^{q} y_k w_k \quad \Longrightarrow \quad \sum_{j=1}^{p} x_j v_j = -\sum_{k=1}^{q} y_k w_k \in \mathcal{M}_1 \cap \mathcal{M}_2$$

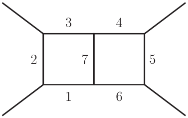
Hence $\sum_{j=1}^{p} x_j v_j$ generate $\mathcal{M}_1 \cap \mathcal{M}_2$, taking $(x_1, \ldots, x_p)$ from each h_i.

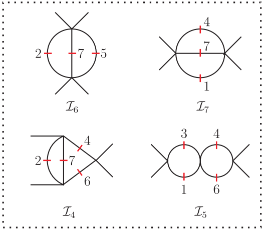
Spanning set of cuts for IBPs

To find the complete IBP reduction, we must consider the cuts associated with "uncollapsible" masters:



A bit more explicitly, the cuts we need to consider are

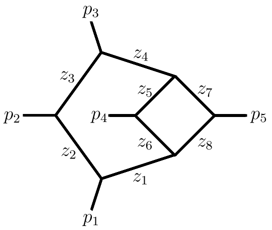


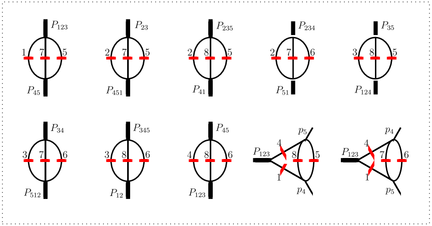


Task: IBP reduce non-planar hexagon box with numerator insertions of degree four in the z_i

[Chicherin, Henn, Mitev JHEP **05**(2018)164]
[S. Badger, C. Brønnum-Hansen, H. Hartanto, T. Peraro, PRL **120**(2018)092001]
[S. Abreu, F. Cordero, H. Ita, B. Page, M. Zeng, 1712.03946]
[H. Chawdhry, M. Lim, A. Mitov, 1805.09182]

There are 10 cuts to consider:



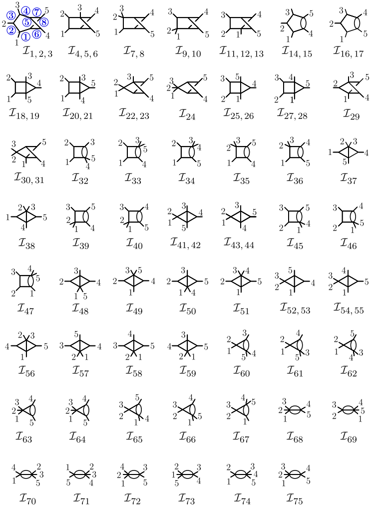


where

$$z_1 = \ell_1^2\,, \qquad z_2 = (\ell_1 - p_1)^2\,, \qquad z_3 = (\ell_1 - P_{12})^2$$
$$z_4 = (\ell_1 - P_{123})^2\,, \qquad z_5 = (\ell_1 + \ell_2 + p_4)^2\,, \qquad z_6 = (\ell_1 + \ell_2)^2$$
$$z_7 = (\ell_2 - p_5)^2\,, \qquad z_8 = \ell_2^2\,, \qquad z_9 = (\ell_1 + p_5)^2$$
$$z_{10} = (\ell_2 + p_1)^2\,, \qquad z_{11} = (\ell_2 + p_2)^2$$

Non-planar hexagon box: integral basis

The spanning set of cuts follows from this list of masters, obtained from AZURITE.



Cut	# of master integrals
$\{1,5,7\}$	26
$\{2,5,7\}$	25
$\{2,5,8\}$	31
$\{2,6,7\}$	31
$\{3,5,8\}$	31
$\{3,6,7\}$	31
$\{3,6,8\}$	25
$\{4,6,8\}$	26
$\{1,4,5,8\}$	13
$\{1,4,6,7\}$	13

Syzygies for the non-planar hexagon box

Syzygies for ensuring D-dimensionality:

$$\begin{aligned}
M_1 = \langle &(z_1 - z_2, z_1 - z_2, -s_{12} + z_1 - z_2, -s_{12} - s_{13} + z_1 - z_2, s_{14} + z_1 - z_2 - z_8 + z_{10}, z_1 - z_2 - z_8 + z_{10}, 0, 0, -s_{12} - s_{13} - s_{14} + z_1 - z_2, 0, 0) \\
&(0,0,0,0, s_{14} + z_1 - z_2 - z_8 + z_{10}, z_1 - z_2 - z_8 + z_{10}, s_{12} + s_{13} + s_{14} - z_8 + z_{10}, z_{10} - z_8, 0, z_{10} - z_8, s_{12} - z_8 + z_{10}) \\
&(s_{12} + z_2 - z_3, z_2 - z_3, z_2 - z_3, -s_{23} + z_2 - z_3, s_{12} + s_{24} + z_2 - z_3 - z_8 + z_{11}, s_{12} + z_2 - z_3 - z_8 + z_{11}, 0, 0, -s_{23} - s_{24} + z_2 - z_3, 0, 0) \\
&(0,0,0,0, s_{12} + s_{24} + z_2 - z_3 - z_8 + z_{11}, s_{12} + z_2 - z_3 - z_8 + z_{11}, s_{12} + s_{23} + s_{24} - z_8 + z_{11}, z_{11} - z_8, 0, s_{12} - z_8 + z_{11}, z_{11} - z_8) \\
&(s_{13} + s_{23} + z_3 - z_4, s_{23} + z_3 - z_4, z_3 - z_4, z_3 - z_4, -2s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_3 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, \\
&\quad -s_{12} + z_3 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, 0, 0, s_{12} + s_{13} + s_{14} + s_{23} + s_{24} + z_3 - z_4, 0, 0) \\
&(0,0,0,0, -2s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_3 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, -s_{12} + z_3 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, \\
&\quad -2s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_4 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, -s_{12} - s_{13} - s_{23} + z_4 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, \\
&\quad 0, -s_{12} - s_{23} + z_4 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, -s_{12} - s_{13} + z_4 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}) \\
&(-s_{12} - s_{13} - s_{23} + z_4 - z_9, -s_{12} - s_{13} - s_{14} - s_{23} + z_4 - z_9, -s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_4 - z_9, z_4 - z_9, z_5 - z_6, z_5 - z_6, 0, 0, z_4 - z_9, 0, 0) \\
&(0,0,0,0, z_5 - z_6, z_5 - z_6, -z_4 + z_5 - z_6 + z_9, s_{12} + s_{13} + s_{23} - z_4 + z_5 - z_6 + z_9, \\
&\quad 0, s_{12} + s_{13} + s_{14} + s_{23} - z_4 + z_5 - z_6 + z_9, s_{12} + s_{13} + s_{23} + s_{24} - z_4 + z_5 - z_6 + z_9) \\
&(2z_1, z_1 + z_2, -s_{12} + z_1 + z_3, -s_{12} - s_{13} - s_{23} + z_1 + z_4, -s_{12} - s_{13} - s_{23} + z_1 + z_4 + z_6 - z_8 - z_9, z_1 + z_6 - z_8, 0, 0, z_1 + z_9, 0, 0) \\
&(0,0,0,0, -s_{12} - s_{13} - s_{23} + z_1 + z_4 + z_6 - z_8 - z_9, z_1 + z_6 - z_8, z_6 - z_8 - z_9, -z_1 + z_6 - z_8, 0, -z_2 + z_6 - z_8, s_{12} - z_1 + z_2 - z_3 + z_6 - z_8) \\
&(-z_1 + z_6 - z_8, -z_1 + z_6 - z_{10}, -z_1 + z_6 + z_8 - z_{10} - z_{11}, s_{12} + s_{13} + s_{23} - z_1 - z_4 + z_5 - z_7 + z_9, \\
&\quad s_{12} + s_{13} + s_{23} - z_1 - z_4 + z_5 + z_8 + z_9, -z_1 + z_6 + z_8, 0, 0, -z_1 + z_6 - z_7, 0, 0) \\
&(0,0,0,0, s_{12} + s_{13} + s_{23} - z_1 - z_4 + z_5 + z_8 + z_9, -z_1 + z_6 + z_8, z_7 + z_8, 2z_8, 0, z_8 + z_{10}, z_8 + z_{11})\rangle \qquad (5.9)
\end{aligned}$$

Syzygies for ensuring no doubled propagators:

$$\begin{aligned}
M_2 = \big\langle &(z_1,0,0,0,0,0,0,0,0,0,0), (0,z_2,0,0,0,0,0,0,0,0,0) \\
&(0,0,z_3,0,0,0,0,0,0,0,0), (0,0,0,z_4,0,0,0,0,0,0,0) \\
&(0,0,0,0,z_5,0,0,0,0,0,0), (0,0,0,0,0,z_6,0,0,0,0,0) \\
&(0,0,0,0,0,0,z_7,0,0,0,0), (0,0,0,0,0,0,0,z_8,0,0,0) \\
&(0,0,0,0,0,0,0,0,1,0,0), (0,0,0,0,0,0,0,0,0,1,0) \\
&(0,0,0,0,0,0,0,0,0,0,1)\big\rangle
\end{aligned}$$

Compute intersection of $M_1\big|_{\text{cut}} \cap M_2\big|_{\text{cut}}$ on each of the 10 cuts.

Resources needed to compute module intersections

Timings and RAM usage $M_1\big|_{\text{cut}} \cap M_2\big|_{\text{cut}}$ on each of the 10 cuts:

cut	time/sec	RAM/GB
$\{1,5,7\}$	218	4.3
$\{2,5,7\}$	43	1.1
$\{2,5,8\}$	303	6.7
$\{2,6,7\}$	743	9.8
$\{3,5,8\}$	404	7.4
$\{3,6,7\}$	699	11.0
$\{3,6,8\}$	24	1.0
$\{4,6,8\}$	797	13.7
$\{1,4,5,8\}$	53	1.7
$\{1,4,6,7\}$	196	3.0

The timings are for an Intel Xeon E5-2643 with 24 cores, 3.40 GHz and 384 GB RAM.

Generators before and after trimming

Trim the initial overcomplete set of generators, i.e. drop the most complicated ones.

Trimming reduces the string sizes by a factor of $\sim$ 2-35.

cut	original size/MB	trimmed size/MB
$\{1,5,7\}$	68	10
$\{2,5,7\}$	25	1.4
$\{2,5,8\}$	49	3.1
$\{2,6,7\}$	100	2.8
$\{3,5,8\}$	97	3.7
$\{3,6,7\}$	80	3.6
$\{3,6,8\}$	10	1.6
$\{4,6,8\}$	21	1.6
$\{1,4,5,8\}$	4.4	3.6
$\{1,4,6,7\}$	9.4	4.1

Plug resulting generators into ansatz for total derivative:

$$0 = \int \left[\sum_{i=1}^{m-c} \left(\frac{\partial a_{r_i}}{\partial z_{r_i}} + \frac{D-L-E-1}{2F(z)} a_{r_i} \frac{\partial F}{\partial z_{r_i}} \right) - \sum_{i=1}^{k-c} \frac{a_{r_i}}{z_{r_i}} \right] \frac{F(z)^{\frac{D-L-E-1}{2}}}{z_{r_1} \cdots z_{r_{k-c}}} \mathrm{d}z_{r_1} \cdots \mathrm{d}z_{r_{m-c}}$$

Complexity of IBP systems

Trim the obtained systems of IBP identities.

The resulting systems take up about $\sim$ 1 MB each and are sparse.

cut	# equations	# integrals	byte size/MB	density
$\{1,5,7\}$	1144	1177	1.2	1.4%
$\{2,5,7\}$	1170	1210	0.99	1.3%
$\{2,5,8\}$	1152	1190	1.1	1.5%
$\{2,6,7\}$	1118	1155	1.0	1.5%
$\{3,5,8\}$	1160	1202	1.2	1.5%
$\{3,6,7\}$	1173	1217	1.3	1.7%
$\{3,6,8\}$	1135	1176	0.77	1.2%
$\{4,6,8\}$	1140	1176	0.94	1.2%
$\{1,4,5,8\}$	700	723	0.69	1.7%
$\{1,4,6,7\}$	683	706	0.66	1.6%

Gauss-Jordan elimination of IBP systems

To find the IBP reductions, Gauss-Jordan eliminate IBP systems.

Some remarks:

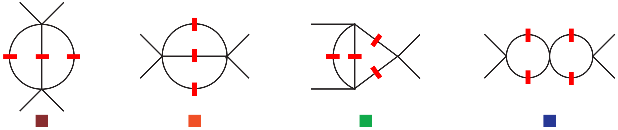
- To preserve sparsity, use a *total pivoting* strategy
  (i.e., allow column swaps)
- For cut $\{1, 4, 6, 7\}$, the RREF can be performed fully analytically, requiring 31 minutes on one core and 1.5 GB RAM.

  ⋮
- For $\{3, 6, 7\}$, assigned numerical values to two s_{ij}.
  Ran 440 points on cluster (2.5 h and 1.8 GB RAM per job).
  Used interpolation code to get analytical results (23 min and 15 GB RAM on one core).

[von Manteuffel and Schabinger, PLB **744**(2015)101]
[Peraro, JHEP**12**(2016)030]

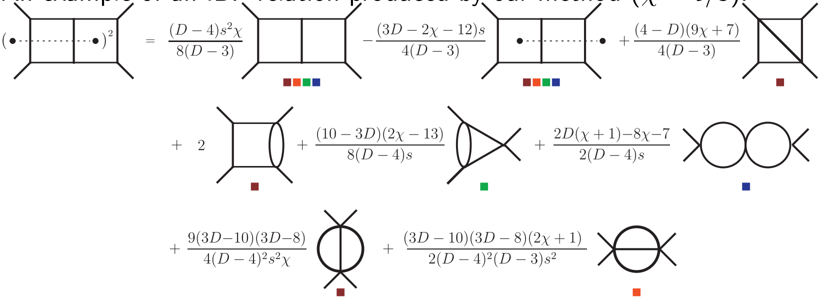
Merging on-shell IBP reductions

By solving the IBP identities on the following cuts



we reconstruct the *complete IBP reductions* by merging the partial results.

An example of an IBP relation produced by our method ($\chi \equiv t/s$):



Results for IBP reductions

- Fully analytic IBP reductions of the 32 hexagon boxes

$$
\begin{array}{lll}
\{I(1,1,1,1,1,1,1,1,0,0,-4), & I(1,1,1,1,1,1,1,1,0,-1,-3), & I(1,1,1,1,1,1,1,1,0,-2,-2) \\
I(1,1,1,1,1,1,1,1,0,-3,-1), & I(1,1,1,1,1,1,1,1,0,-4,0), & I(1,1,1,1,1,1,1,1,-1,0,-3) \\
I(1,1,1,1,1,1,1,1,-1,-1,-2), & I(1,1,1,1,1,1,1,1,-1,-2,-1), & I(1,1,1,1,1,1,1,1,-1,-3,0) \\
I(1,1,1,1,1,1,1,1,-2,0,-2), & I(1,1,1,1,1,1,1,1,-2,-1,-1), & I(1,1,1,1,1,1,1,1,-2,-2,0) \\
I(1,1,1,1,1,1,1,1,-3,0,-1), & I(1,1,1,1,1,1,1,1,-3,-1,0), & I(1,1,1,1,1,1,1,1,-4,0,0) \\
I(1,1,1,1,1,1,1,1,0,0,-3), & I(1,1,1,1,1,1,1,1,0,-1,-2), & I(1,1,1,1,1,1,1,1,0,-2,-1) \\
I(1,1,1,1,1,1,1,1,0,-3,0), & I(1,1,1,1,1,1,1,1,-1,0,-2), & I(1,1,1,1,1,1,1,1,-1,-1,-1) \\
I(1,1,1,1,1,1,1,1,-1,-2,0), & I(1,1,1,1,1,1,1,1,-2,0,-1), & I(1,1,1,1,1,1,1,1,-2,-1,0) \\
I(1,1,1,1,1,1,1,1,-3,0,0), & I(1,1,1,1,1,1,1,1,0,0,-2), & I(1,1,1,1,1,1,1,1,0,-1,-1) \\
I(1,1,1,1,1,1,1,1,0,-2,0), & I(1,1,1,1,1,1,1,1,-1,0,-1), & I(1,1,1,1,1,1,1,1,-1,-1,0) \\
I(1,1,1,1,1,1,1,1,0,0,-1), & I(1,1,1,1,1,1,1,1,0,-1,0)\} &
\end{array}
$$

  can be downloaded from (268 MB compressed / 790 MB uncompressed)

  https://github.com/yzhphy/hexagonbox_reduction/releases/download/1.0.0/hexagon_box_degree_4_Final.zip

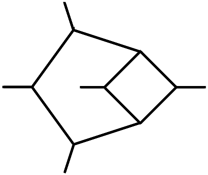
- Our results agree with fully numerical results from FIRE5 C++ (6 hours per point).

  [A. Smirnov, CPC **189**(2015)182]

Conclusions

- New formalism for IBP reductions. Main ideas: Baikov rep., cuts, syzygies, module intersection algorithms, total pivoting, rational reconstruction

- Obtained the fully analytic IBP reductions of



with numerator insertions up to degree 4 in the z_i.

- Powerful framework. IBP reductions for further $2 \to 3$ two-loop processes seem well within reach.