Quantum Field Theory II

Problem Sets

ETH Zurich, 2026 FS

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The current version of this work can be found at:
http://people.phys.ethz.ch/~nbeisert/lectures/.

Contents

1.1. Transition amplitude for the harmonic oscillator

A standard exercise regarding path integrals in quantum mechanics is the computation of the transition amplitude for a harmonic oscillator. The harmonic oscillator is specified by the Lagrange function

$$L(q, \dot{q}) = \tfrac{1}{2} m \dot{q}^2 - \tfrac{1}{2} m \omega^2 q^2, \tag{1.1}$$

where q and $\dot{q}$ are the position and velocity variables. In terms of the path integral in position space, the transition amplitude is given by

$$U(q_\mathrm{f}, q_\mathrm{i}, t) := \langle q_\mathrm{f}, t_\mathrm{f} | q_\mathrm{i}, t_\mathrm{i} \rangle = \int \mathrm{D}q \exp\left[\frac{\mathring{\imath}}{\hbar} \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\, L(q, \dot{q})\right], \tag{1.2}$$

where $t := t_\mathrm{f} - t_\mathrm{i}$. In this guided exercise, you show that it equals

$$U(q_\mathrm{f}, q_\mathrm{i}, t) = \left[\frac{m\omega}{2\pi\hbar\mathring{\imath}\sin(\omega t)}\right]^{1/2} \exp\left[\frac{\mathring{\imath} m\omega}{\hbar} \frac{\frac{1}{2}(q_\mathrm{i}^2 + q_\mathrm{f}^2)\cos(\omega t) - q_\mathrm{i} q_\mathrm{f}}{\sin(\omega t)}\right]. \tag{1.3}$$

a) To get started, we split up the path $q(t)$ in n intermediate steps q_k, $k = 1, \ldots, n-1$, at equally spaced times. In the lecture we have seen that this procedure leads us to the discretised expression

$$U(q_\mathrm{f}, q_\mathrm{i}, t) = \lim_{n\to\infty} \int \left[\prod_{j=1}^{n-1} \mathrm{d}q_j\right] \left[\frac{nm}{2\pi\hbar\mathring{\imath} t}\right]^{n/2} \exp\left[\frac{\mathring{\imath}}{\hbar} \sum_{k=1}^{n} \frac{t}{n} L_k\right], \tag{1.4}$$

where $q_0 := q_\mathrm{i}$, $q_n := q_\mathrm{f}$ and

$$L_k = \frac{m}{2}\left(\frac{q_k - q_{k-1}}{t/n}\right)^2 - V(q_k). \tag{1.5}$$

In order to compute the integral, rewrite the exponential in (1.4) as a Gaussian function of the form

$$\exp\left[\frac{\mathring{\imath}}{\hbar} \sum_{k=1}^{n} \frac{t}{n} L_k\right] = \exp\left[\frac{\mathring{\imath} nm}{\hbar t}\left(\tfrac{1}{2}\vec{q}^{\mathsf{T}} M \vec{q} + \vec{B}^{\mathsf{T}} \vec{q} + C\right)\right], \tag{1.6}$$

where $\vec{q} = (q_1, q_2, \ldots, q_{n-1})$ is an $(n-1)$-dimensional vector, M is an $(n-1)\times(n-1)$ matrix, $\vec{B}$ is an $(n-1)$-dimensional vector and C a constant (the latter two depending on $q_0 = q_\mathrm{i}$, $q_n = q_\mathrm{f}$). With this form, you should be able to compute the integral (rather) easily.

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b) Next, we turn to the computation of the determinant of M. In order to do this, consider the minor M_j of M of size $j \times j$ with the last $n-1-j$ rows and columns eliminated, i.e. the upper left $(j \times j)$-block of M. Show that the determinant of $D_j := \det M_j$ satisfies the equality

$$\frac{D_{j+1} - 2D_j + D_{j-1}}{(t/n)^2} = -a^2 D_j. \tag{1.7}$$

Determine the constant a, then solve the equation.

Hint: This difference equation is the discretised version of a well-known differential equation. In order to solve it, you can solve the continuum equations for $D(\tau)$, re-express the solution in discretised time and match the coefficients to some small values of j. You should find

$$D_j \simeq \frac{n}{\omega t} \sin(\omega t) + (j - n + 1) \cos(\omega t) + \mathcal{O}(1/n) \quad \text{for } j \approx n \to \infty. \tag{1.8}$$

c) Now you are left with the computation of the coefficients of q_i^2, q_f^2 and $q_\mathrm{i} q_\mathrm{f}$ in the exponent. You can compute them by computing the appropriate minors of M and get the correct result. Enjoy!

2.1. Propagator of a free Klein–Gordon field

In this problem we will derive the Feynman propagator of a free scalar field using the path integral formalism. To that end, we will frequently need some generic Gaußian integrals.

a) Compute the Gaußian integrals

$$\int \mathrm{d}x^n \exp\bigl(-\tfrac{1}{2}x^{\mathsf{T}}Mx\bigr) \qquad \text{and} \qquad \int \mathrm{d}x^n\, x_j x_k \exp\bigl(-\tfrac{1}{2}x^{\mathsf{T}}Mx\bigr), \tag{2.1}$$

where x is an n-dimensional vector and M is a symmetric $n\times n$ matrix.

Hint: For the first integral, diagonalise M by a change of variables. Express the prefactor of the second integral using a derivative w.r.t. a suitable element of M.

To compute the relevant path integral

$$\begin{aligned}\langle 0|\,\mathrm{T}[\phi(x_1)\phi(x_2)]|0\rangle &= \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathrm{D}\phi\,\phi(x_1)\phi(x_2)\exp\bigl(i\int_{-T}^{+T}\mathrm{d}x^4\,\mathcal{L}\bigr)}{\int \mathrm{D}\phi\exp\bigl(i\int_{-T}^{+T}\mathrm{d}x^4\,\mathcal{L}\bigr)} \\ &= \int \frac{\mathrm{d}k^4}{(2\pi)^4}\,\frac{-i\,\mathrm{e}^{-ik(x_1-x_2)}}{k^2+m^2-i\epsilon}\,,\end{aligned} \tag{2.2}$$

we discretise spacetime by considering a four-dimensional lattice with N lattice points per side of length L. In the continuum limit, the lattice spacing L/N goes to zero, and the lattice size L goes to infinity. We perform a discrete mode expansion of the free scalar field

$$\phi(x) = \frac{1}{L^4}\sum_n \mathrm{e}^{-ik_n x}\,\phi(k_n), \tag{2.3}$$

where $k_n^\mu = 2\pi n^\mu/L$ is the discretised momentum, with $-N/2 \le n^\mu \le N/2$ an integer. The individual Fourier coefficients $\phi(k_n)$ are complex but the field $\phi(x)$ is real, so that $\phi(k_n)^* = \phi(-k_n)$. However, we can treat the real and imaginary part $\phi_\mathrm{r}(k), \phi_\mathrm{i}(k)$ of $\phi(k)$ as independent variables if we restrict ourselves to modes with $k_n^0 > 0$. Finally, the integrals are replaced by (the range '$n,+$' indicates the restriction to modes with $k_n^0 > 0$)

$$\int \mathrm{D}\phi \to \int \prod_{n,+}\bigl(\mathrm{d}\phi_\mathrm{r}(k_n)\,\mathrm{d}\phi_\mathrm{i}(k_n)\bigr), \qquad \int \frac{\mathrm{d}k^4}{(2\pi)^4} \to \frac{1}{L^4}\sum_n\,. \tag{2.4}$$

b) Find the discretised equivalent to the action of the Klein–Gordon field

$$S = \int \mathrm{d}x^4\,\bigl(-\tfrac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \tfrac{1}{2}m^2\phi(x)^2\bigr). \tag{2.5}$$

Hint: Use reality of $\phi(x)$ to express the result quadratically in $\phi_\mathrm{r}(k_n)$ and $\phi_\mathrm{i}(k_n)$.

c) Making use of the results obtained in part a), compute the discrete equivalent of

$$\int \mathrm{D}\phi\,\mathrm{e}^{iS}\,. \tag{2.6}$$

d) Now compute the discretised version of (2.2) by inserting the mode expansion for $\phi(x_1)\phi(x_2)$. Make use of the symmetry of the integrand (even or odd in ϕ) to maintain only non-vanishing terms. Can you relate your expressions to what you found in part a)? Finally, take the continuum limit and recover the Feynman propagator.

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2.2. Generating functional

Consider a Lagrangian for scalars with a quartic interaction

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \qquad \mathcal{L}_0 = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2, \qquad \mathcal{L}_{\text{int}}(\phi) = -\tfrac{1}{24}\lambda\phi^4. \tag{2.7}$$

From the lecture you know that the interacting generating functional is then

$$Z[j] = \exp\left[\mathring{i}\int \mathrm{d}x^D\,\mathcal{L}_{\text{int}}\left(\frac{-\mathring{i}\delta}{\delta j(x)}\right)\right]Z_0[j], \qquad Z_0[j] = \int \mathrm{D}\phi\exp\left[\mathring{i}\int \mathrm{d}x^D\left(\mathcal{L}_0 + j(x)\,\phi(x)\right)\right]. \tag{2.8}$$

a) Show that

$$Z_0[j] = Z_0[0]\exp\left[\frac{\mathring{i}}{2}\int \mathrm{d}x^D\,\mathrm{d}y^D\,j(x)\,G_{\mathrm{F}}(x,y)\,j(y)\right], \tag{2.9}$$

and argue that $G_{\mathrm{F}}(x,y)$ is indeed the *Feynman* propagator.

b) Compute the vacuum contributions to $Z[j]$ to order λ^2,

$$\begin{aligned}\frac{Z[0]}{Z_0[0]} &= \frac{1}{Z_0[0]}\exp\left[\mathring{i}\int \mathrm{d}x^D\,\mathcal{L}_{\text{int}}\left(\frac{-\mathring{i}\delta}{\delta j(x)}\right)\right]Z_0[j]\bigg|_{j=0}\\ &= 1 + \mathring{i}\lambda\int \mathrm{d}x^D\,C_1G_{\mathrm{F}}(x,x)^2 + \lambda^2\int \mathrm{d}x^D\,\mathrm{d}y^D\,\Big[C_{2,1}\,G_{\mathrm{F}}(x,x)^2G_{\mathrm{F}}(y,y)^2\\ &\qquad + C_{2,2}G_{\mathrm{F}}(x,x)G_{\mathrm{F}}(y,y)G_{\mathrm{F}}(x,y)^2 + C_{2,3}\,G_{\mathrm{F}}(x,y)^4\Big] + \mathcal{O}(\lambda^3),\end{aligned} \tag{2.10}$$

being particularly careful in the computation of the combinatorial factors C_1, $C_{2,1}$, $C_{2,2}$, $C_{2,3}$. These factors can be computed either from the functional derivative expression or as symmetry factors of the related diagrams.

Describe graphically each term and identify connected and disconnected terms.

Quantum Field Theory II — Problem Set 3

ETH Zurich, 2026 FS — Prof. N. Beisert

due: 11.03.2026

3.1. Disconnected and connected functionals

Consider now the Lagrangian of a massive scalar field ϕ with quartic interaction

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \qquad \mathcal{L}_0 = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2, \qquad \mathcal{L}_{\text{int}} = -\tfrac{1}{24}\lambda\phi^4. \tag{3.1}$$

The generating functional with sources $j(x)$ is then

$$Z[j] = Z_0[0]\exp\left[\mathring{\imath}\int \mathrm{d}x^4\,\mathcal{L}_{\text{int}}\left(\frac{-\mathring{\imath}\delta}{\delta j(x)}\right)\right]\exp\left[\frac{\mathring{\imath}}{2}\int \mathrm{d}y_1^4\,\mathrm{d}y_2^4\,j(y_1)\,G_{\text{F}}(y_1-y_2)\,j(y_2)\right], \tag{3.2}$$

where $G_{\text{F}}(x)$ is the Feynman propagator. Recall that $Z[j]$ includes vacuum contributions. Furthermore, recall that the functional

$$W[j] = -\mathring{\imath}\log Z[j] \tag{3.3}$$

generates only the connected graphs.

a) Compute $Z[j]$ and $W[j]$ up to and including orders λ and j^4, i.e. drop any contributions with more than one vertex or more than four sources.

Hint: The entire exercise can be solved graphically.

b) Convince yourself graphically that the contributions to the four-point correlation function at order λ arise correctly from

$$\big\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\big\rangle = Z[j]^{-1}\,\frac{-\mathring{\imath}\delta}{\delta j(x_1)}\,\frac{-\mathring{\imath}\delta}{\delta j(x_2)}\,\frac{-\mathring{\imath}\delta}{\delta j(x_3)}\,\frac{-\mathring{\imath}\delta}{\delta j(x_4)}\,Z[j]\bigg|_{j=0} \tag{3.4}$$

and that the connected contributions arise from

$$\frac{-\mathring{\imath}\delta}{\delta j(x_1)}\,\frac{-\mathring{\imath}\delta}{\delta j(x_2)}\,\frac{-\mathring{\imath}\delta}{\delta j(x_3)}\,\frac{-\mathring{\imath}\delta}{\delta j(x_4)}\,\mathring{\imath}W[j]\bigg|_{j=0}. \tag{3.5}$$

c) Determine the connected contributions at order λ^2 with four external legs.

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3.2. Classical field configurations

Consider again the Lagrangian for ϕ^4-model

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \qquad \mathcal{L}_0 = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2, \qquad \mathcal{L}_{\text{int}} = -\tfrac{1}{24}\lambda\phi^4 \tag{3.6}$$

and its generating functional $Z[j]$ in the path integral

$$Z[j] = \int \mathrm{D}\phi \exp \mathring{\imath} \biggl[S[\phi] + \int \mathrm{d}x^D\, \phi(x) j(x)\biggr]. \tag{3.7}$$

The generating functional $Z[j]$ may be approximated via the method of stationary phase. Namely, if $\phi_{\mathrm{C}}[j]$ is the field configuration where the functional derivative of the exponent vanishes,

$$\frac{\delta S}{\delta\phi}\bigl[\phi_{\mathrm{C}}[j]\bigr] := \frac{\delta S[\phi]}{\delta\phi}\bigg|_{\phi=\phi_{\mathrm{C}}[j]} = -j, \tag{3.8}$$

the path integral is approximated by evaluating the exponent at this point (up to an irrelevant factor independent of j)

$$Z[j] \simeq \exp \mathring{\imath} \biggl[S\bigl[\phi_{\mathrm{C}}[j]\bigr] + \int \mathrm{d}x^D\, \phi_{\mathrm{C}}[j](x)\, j(x)\biggr]. \tag{3.9}$$

For the connected functional, this implies

$$W[j] = -\mathring{\imath} \log Z[j] \simeq S\bigl[\phi_{\mathrm{C}}[j]\bigr] + \int \mathrm{d}x^D\, \phi_{\mathrm{C}}[j](x)\, j(x) =: T[j]. \tag{3.10}$$

In the following, use algebraic equations to obtain your answers, then give a graphical representation of results.

a) Compute the *classical field configuration* $\phi_{\mathrm{C}}[j]$ that solves the *classical equation of motion* (3.8) using perturbation theory up to and including order λ^2.

b) Compute the two-point correlation function that stems from $\mathring{\imath}T[j]$ by taking functional derivatives. What can be stated about higher-order perturbative corrections to this result?

Quantum Field Theory II

Problem Set 4

ETH Zurich, 2026 FS

Prof. N. Beisert

due: 18. 03. 2026

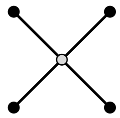
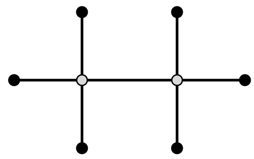
4.1. Effective action

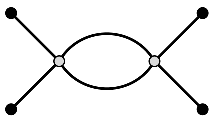
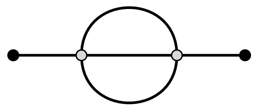
Here we graphically compute the effective action $G[\phi]$ for the model defined by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \qquad \mathcal{L}_0 = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2, \qquad \mathcal{L}_{\text{int}} = -\tfrac{1}{24}\lambda\phi^4 \tag{4.1}$$

at $\mathcal{O}(\lambda^2)$ ignoring vacuum bubbles and tadpoles.

You may start with the connected functional $W[j]$ given by the graphs

$\mathring{\imath}W[j] = \dfrac{1}{2}$ $+\ \dfrac{\lambda}{24}$  $+\ \dfrac{\lambda^2}{72}$ 

$+\ \dfrac{\lambda^2}{16}$  $+\ \dfrac{\lambda^2}{12}$  $+\ \ldots.$ (4.2)

Throughout the calculation you should ignore any vacuum bubble graph, tadpole graph and graph with more than two interaction vertices.

a) Compute the field functional $\phi[j] = \delta W[j]/\delta j$.

b) Determine the inverse functional $j[\phi]$ by demanding that $j[\phi[j]] = j$ or $\phi[j[\phi]] = \phi$.
Hint: Use the result you obtained in part a). Substitute graphically the result you obtained at leading order in λ into the all higher order terms, and continue like this order by order in λ. Immediately drop terms of order λ^3.

c) Compute $W[j[\phi]]$ and $S_{\text{src}}[\phi, j[\phi]] = \int \mathrm{d}x^4\, \phi(x) j[\phi](x)$ by graphically substituting the result you obtained for $j[\phi]$ in part b).

d) Take the difference $G[\phi] = W[j[\phi]] - S_{\text{src}}[\phi, j[\phi]]$. Convince yourself that all remaining terms in $G[\phi]$ are 1PI.

4.2. Chromodynamics Lagrangian

The chromodynamics Lagrangian is given by

$$\mathcal{L} = \bar{\psi}(\gamma^\mu D_\mu - m)\psi - \frac{1}{2g^2}\operatorname{tr}(F^{\mu\nu}F_{\mu\nu}), \tag{4.3}$$

where $F_{\mu\nu}$ is the field strength tensor of a SU(3) gauge field A_μ and ψ is a 3-vector of Dirac fields transforming in the defining representation of SU(3).

a) Write down the equations of motion for the fields.

b) Show that the current $(J^\mu)^\alpha{}_\beta = \mathring{\imath}\bar{\psi}_\beta\gamma^\mu\psi^\alpha$ is covariantly conserved: $[D_\mu, J^\mu] = 0$.

c) Compose a gauge-invariant Lagrangian for a scalar field and a corresponding current.

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4.3. Wilson lines and loops

In this problem we discuss Wilson lines and Wilson loops as gauge-covariant expressions of the gauge field. In terms of differential geometry, it is also know as the parallel transport.

Let us first consider the abelian case. The *Wilson line* for an open path γ is defined by

$$W[\gamma] = \exp\left[\mathring{\imath}\int_\gamma A\right] = \exp\left[\mathring{\imath}\int \mathrm{d}\tau\, \dot{y}^\mu(\tau) A_\mu\big(y(\tau)\big)\right]. \tag{4.4}$$

Here, $A(x) := \mathrm{d}x^\mu\, A_\mu(x)$ is a one-form, the so-called *gauge connection*, and $\gamma : \tau \mapsto y^\mu(\tau)$ is a parametrisation of γ on some interval for τ with $\dot{y}^\mu := \mathrm{d}y^\mu/\mathrm{d}\tau$. For a closed path γ, $W[\gamma]$ is called a *Wilson loop*.

a) Using Stokes' theorem, rewrite an abelian Wilson loop $W[\gamma]$ in terms of the field strength $F = \frac{1}{2}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\, F_{\mu\nu}$. Conclude that this Wilson loop is gauge invariant.

b) Consider a fixed path γ and denote by $\gamma_{t,s}$ the sub-path on the interval $\tau \in [s,t]$. Show that $W[\gamma_{t,s}]$ satisfies the differential equations

$$\begin{aligned}\partial_t W[\gamma_{t,s}] &= +\mathring{\imath}\dot{y}^\mu(t)\, A_\mu\big(y(t)\big)\, W[\gamma_{t,s}],\\ \partial_s W[\gamma_{t,s}] &= -\mathring{\imath} W[\gamma_{t,s}]\, \dot{y}^\mu(s)\, A_\mu\big(y(s)\big).\end{aligned} \tag{4.5}$$

Consider now a non-abelian, matrix-valued gauge field for which the Wilson line is given by

$$W[\gamma_{t,s}] = \overleftarrow{\mathrm{P}}\exp\left[\mathring{\imath}\int_s^t \mathrm{d}\tau\, \dot{y}^\mu(\tau) A_\mu\big(y(\tau)\big)\right]. \tag{4.6}$$

Here, $\overleftarrow{\mathrm{P}}$ denotes path ordering which means that the terms in the expansion of the exponential are ordered in such a way that higher values of τ stand to the left.

c) Show that the non-abelian Wilson line satisfies the same differential equations as in the abelian case paying close attention to the particular ordering used in (4.5).

d) Show that $W[\gamma_{t,s}]$ transforms under a gauge transformation given by $U(x)$ as

$$W'[\gamma_{t,s}] = U\big(y(t)\big)\, W[\gamma_{t,s}]\, U\big(y(s)\big)^{-1}. \tag{4.7}$$

Hint: Consider an infinitesimal gauge transformation and note that a Wilson loop transforms as $\delta W[\gamma_{t,\tau}] = \mathring{\imath}\int \mathrm{d}\tau\, W[\gamma_{t,\tau}]\, \dot{y}^\mu(\tau)\, \delta A_\mu(y(\tau))\, W[\gamma_{\tau,s}]$.

e) We can construct a Wilson loop $W(x,m,n)$ using a small rectangular path about a point x as

$$\begin{aligned}W(x,m,n) := {}& W(x, x+\epsilon n)\, W(x+\epsilon n, x+\epsilon n+\epsilon m)\\ &\cdot W(x+\epsilon n+\epsilon m, x+\epsilon m)\, W(x+\epsilon m, x),\end{aligned} \tag{4.8}$$

where we have denoted the Wilson line connecting two points x, y with a straight path γ by $W(y,x)$. Expand W to second order in ϵ and compare to the field strength $F_{\mu\nu} = \mathring{\imath}[D_\mu, D_\nu]$. Also compare your result with part a) for the corresponding path.

Quantum Field Theory II — Problem Set 5

ETH Zurich, 2026 FS — Prof. N. Beisert

due: 25. 03. 2026

5.1. Gauge fixing in the path integral

a) Consider pure electrodynamics. Perform gauge fixing via the Faddeev–Popov method using the non-linear gauge condition

$$G[A, \Omega] = \partial^\mu A_\mu + \tfrac{1}{2}\zeta A^\mu A_\mu - \Omega. \tag{5.1}$$

Invert the kinetic operators that appear in the action to find the propagators for the photon field and for the ghost field.

Is the ghost field decoupled in this gauge? Show that in the limit $\zeta \to 0$ the R_ξ-gauge is restored.

b) Consider now pure Yang–Mills theory. Perform the gauge fixing via the Faddeev–Popov method using the axial gauge condition along a fixed four-vector n^μ

$$G[A, \Omega] = n^\mu A_\mu - \Omega. \tag{5.2}$$

Invert the kinetic operators that appear in the action to find the propagators for the ghost field and for the gluon field.

Under which condition do ghosts decouple from gluon fields in this gauge? How can this be exploited in practice?

5.2. Cohomology for null vectors in Minkowski space

Consider Minkowski space $\mathbb{V} := \mathbb{R}^{D-1,1}$ with a fixed null vector n^μ and the operator

$$Qv := (n{\cdot}v)n. \tag{5.3}$$

a) Show that the operator Q squares to zero if and only if n is null.

Having an operator that squares to zero allows us to organise $\mathbb{V}$ in terms of the cohomology of Q. Define the following two subspaces and quotient spaces:

$$\mathbb{V}_{\text{cl}} := \ker Q, \qquad \mathbb{V}_{\text{ex}} := \operatorname{im} Q, \qquad \mathbb{H}_{\text{phys}} := \mathbb{V}_{\text{cl}}/\mathbb{V}_{\text{ex}}, \qquad \mathbb{H}_{\text{op}} := \mathbb{V}/\mathbb{V}_{\text{cl}}. \tag{5.4}$$

The kernel and image of Q are known as closed and exact subspaces, the quotient of $\mathbb{H}_{\text{phys}}$ is the cohomology of Q, and $\mathbb{H}_{\text{op}}$ is the complement of the closed subspace.

b) What are the dimensions of these subspaces? Show that $\mathbb{V}_{\text{ex}} \subset \mathbb{V}_{\text{cl}}$; why is this relation relevant? Describe the sets $\mathbb{H}_{\text{phys}}$ and $\mathbb{H}_{\text{op}}$ in terms of their elements. Argue that $\mathbb{H}_{\text{op}}$ is isomorphic to $\mathbb{V}_{\text{ex}}$.

c) Argue that $\mathbb{V} = \mathbb{V}_{\text{ex}} \oplus \mathbb{H}_{\text{phys}} \oplus \mathbb{H}_{\text{op}}$. Choose a concrete basis c_j^μ, $j = 1, \ldots, 4$ for $\mathbb{V}$ such that the three subspaces $\mathbb{V}_{\text{ex}}$, $\mathbb{H}_{\text{phys}}$ and $\mathbb{H}_{\text{op}}$ are spanned by subsets of it.

Hint: There are tame and wild choices. All all equally valid; also try a wilder one.

d) Construct the dual basis $\tilde{c}_j^\mu$, $j = 1, \ldots, 4$ for $\mathbb{V}$ with respect to the Minkowski metric. In other words, the dual basis must satisfy $c_j{\cdot}\tilde{c}_k = \delta_{j,k}$.

e) Give a decomposition of unity in terms of the subspaces found above.

f) Finally, show that the Minkowski metric reduces on $\mathbb{H}_{\text{phys}}$ to a positive definite one.

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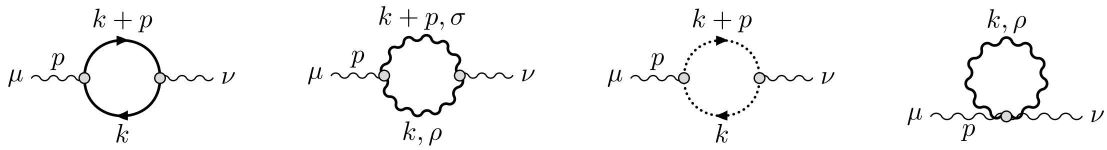
5.3. Transversality of the gluon two-point function

In this exercise, we consider the one-loop gluon two-point function $M^{(1)}_{\mu\nu}$ in Yang–Mills theory coupled to a spinor matter field. Quantising Yang–Mills theory requires the introduction of ghosts, which will contribute to the gluon two-point function. We will show that this contribution is essential for the gluon two-point function to retain transversality, $p^\mu M^{(1)}_{\mu\nu}(p) = 0$. For convenience, we will ignore all aspects of the matrix-valued fields and focus exclusively on the momentum dependency.

The one-loop gluon two-points function receives contribution from four diagrams which will be discussed one-by-one in the following four parts:



(5.5)

a) Consider first the spinor loop contribution $M^{\text{qu}}_{\mu\nu}$. As we only care about the momentum dependency of this diagram, we can use the Feynman rules for QED. Compute $M^{\text{qu}}_{\mu\nu}(p)$ as a function of the external momentum p^μ. Then, check that this contribution is properly transverse, $p^\mu M^{\text{qu}}_{\mu\nu}(p) = 0$.

Hint: Do not try to evaluate the momentum integrals you will encounter. It suffices to use reparametrisation properties of the integrals.

b) Now, consider the gluon loop with two cubic vertices. The momentum dependency of this amplitude reads

$$M^{\text{gl}}_{\mu\nu} = \frac{1}{2} \int \frac{\mathrm{d}k^4}{(2\pi)^4} \frac{F^{\rho\sigma}{}_\mu(k, -k-p) F_{\sigma\rho\nu}(k+p, -k)}{k^2(k+p)^2} \tag{5.6}$$

with the momentum dependency of the cubic gluon vertex given by

$$F_{\rho\sigma\mu}(p_1, p_2) := \eta_{\rho\sigma}(p_1 - p_2)_\mu + \eta_{\sigma\mu}(p_1 + 2p_2)_\rho + \eta_{\mu\rho}(-2p_1 - p_2)_\sigma. \tag{5.7}$$

Compute $p^\mu M^{\text{gl}}_{\mu\nu}$ and show that it does not vanish on its own.

Hint: It is convenient to drop all appearances of the tadpole loop integral $\int \mathrm{d}k^4/k^2$ right away; we shall consider them later in part d).

c) Now, consider the ghost loop. Its momentum dependency reads

$$M^{\text{gh}}_{\mu\nu} = -\int \frac{\mathrm{d}k^4}{(2\pi)^4} \frac{k^\mu (k+p)^\nu}{k^2(k+p)^2}\,. \tag{5.8}$$

Compute $p^\mu M^{\text{gh}}_{\mu\nu}$. Show that $p^\mu (M^{\text{gl}}_{\mu\nu} + M^{\text{gh}}_{\mu\nu}) = 0$ up to tadpole loop integrals $\int \mathrm{d}k^4/k^2$.

Hint: For integrals of the form $\int \mathrm{d}k^4\, k_\mu/k^2(p+k)^2$, you may use the transformation $k \to -p-k$ to simplify the numerator.

d) *bonus:* Finally consider the gluon tadpole graph whose momentum dependency reads

$$M^{\text{tad}}_{\mu\nu} = -G_{\mu\rho,\nu}{}^{\rho} \int \frac{\mathrm{d}k^4}{(2\pi)^4} \frac{1}{k^2} \qquad \text{with} \qquad G_{\mu\nu,\rho\sigma} := \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}. \tag{5.9}$$

Show that it makes the one-loop gluon two-point function properly transversal

$$p_\mu (M^{\text{gl}}_{\mu\nu} + M^{\text{gh}}_{\mu\nu} + M^{\text{tad}}_{\mu\nu}) = 0. \tag{5.10}$$

Quantum Field Theory II

Problem Set 6

ETH Zurich, 2026 FS

Prof. N. Beisert

due: 15. 04. 2026

6.1. Passarino–Veltman reduction

We will study a method to reduce one-loop tensor integrals to linear combinations of scalar integrals. For convenience, the integrals are defined in D spatial dimensions with Wick-rotated momenta. Specifically, we define the integrals as

$$A_0(m^2) := \int \frac{\mathrm{d}k^D}{(2\pi)^D} \frac{1}{k^2+m^2}\,, \tag{6.1}$$

$$B_{0,\mu,\mu\nu} := \int \frac{\mathrm{d}k^D}{(2\pi)^D} \frac{(1, k_\mu, k_\mu k_\nu)}{[k^2+m_1^2][(k-p)^2+m_2^2]}\,. \tag{6.2}$$

For simplicity, we shall suppress the arguments $(p; m_1^2, m_2^2)$ of the functions B.

The goal is to reduce the integrals $B_\mu, B_{\mu\nu}$ to a linear combination of the scalar integrals A_0, B_0 with suitable coefficient functions. The ansatz is to decompose the tensor integrals according to their index structure; the allowed structures for the two tensor integrals are

$$B_\mu = p_\mu B_1, \tag{6.3}$$

$$B_{\mu\nu} = p_\mu p_\nu B_{21} + \eta_{\mu\nu} B_{22}, \tag{6.4}$$

where B_1, B_{21}, B_{22} are linear combinations of A_0 and B_0 which are to be determined.

a) Show that B_1 in (6.3) can be expressed as the following linear combination of $A_0(m_1^2)$, $A_0(m_2^2)$ and B_0:

$$B_1 = -\frac{1}{2p^2} A_0(m_1^2) + \frac{1}{2p^2} A_0(m_2^2) + \frac{p^2+m_2^2-m_1^2}{2p^2} B_0. \tag{6.5}$$

Hint: Contract both sides of (6.3) with p^μ and rewrite the k-dependence in the numerator of the integrand as a linear combination of the two factors in the denominator.

b) For the tensor integral $B_{\mu\nu}$, we can obtain a 2×2 linear system of equations by contracting both sides of (6.4) with $\eta^{\mu\nu}$ and $p^\mu p^\nu$, respectively.

Repeat the steps for the manipulation of the numerator of the integrands. Show that the linear system expressing B_{21}, B_{22} in terms of the scalar integrals A_0, B_0 is

$$\begin{aligned} p^2 B_{21} + D B_{22} &= X_1 A_0(m_2^2) + X_2 B_0, \\ p^2 B_{21} + B_{22} &= Y_1 A_0(m_2^2) + Y_2 B_1, \end{aligned} \tag{6.6}$$

where X_i, Y_i are some functions to be determined. Finally, solve for B_{21}, B_{22}.

Hint: You may recycle the results of part a); furthermore, recall the integral

$$\int \frac{\mathrm{d}k^D}{(2\pi)^D} \frac{k^\mu}{k^2+m^2} = 0. \tag{6.7}$$

$\longrightarrow$

6.2. Feynman's parametrisation of loop integrals

Prove the following identity, which is used to combine the denominators of integrands in loop integrals

$$\frac{1}{D_1^{a_1}\dots D_n^{a_n}} = \frac{\Gamma(\sum_{j=1}^n a_j)}{\Gamma(a_1)\dots\Gamma(a_n)}\int_0^1 \mathrm{d}x_1\dots\int_0^1 \mathrm{d}x_n\,\frac{\delta(1-\sum_{j=1}^n x_j)x_1^{a_1-1}\dots x_n^{a_n-1}}{[x_1D_1+\dots+x_nD_n]^{a_1+\dots+a_n}}\,. \tag{6.8}$$

Hint: Start with

$$\frac{\Gamma(a)}{D^a} = \int_0^\infty \mathrm{d}t\, t^{a-1}\,\mathrm{e}^{-tD}\,, \tag{6.9}$$

which defines the gamma-function. Put an index on D, α, and t, and take the product. Then multiply on the right-hand side by

$$1 = \int_0^\infty \mathrm{d}s\,\delta\big(s-\textstyle\sum_i t_i\big). \tag{6.10}$$

Make the change of variable $t_i = sx_i$ and carry out the integral over s.

6.3. Renormalisation of cubic scalar theory in 6D

Consider a scalar theory in $D=6$ with cubic interaction

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu\phi_0\,\partial^\mu\phi_0 - \tfrac{1}{2}m_0^2\phi_0^2 - \tfrac{1}{6}\lambda_0\phi_0^3. \tag{6.11}$$

In this problem we want to derive the one-loop renormalisation of the bare couplings in dimensional regularisation at $D=6-2\epsilon$.

a) Show, by power counting, that the theory is renormalisable in $D=6$.

b) Rewrite the Lagrangian in terms of the renormalised field $\phi=\kappa^{-1}\phi_0$. Then adjust the interaction term by a suitable power of the scale parameter μ for dimensional regularisation such that the coupling constant λ_0 becomes dimensionless.

c) Derive the Feynman rules for the theory.

d) Compute the one-loop self energy contribution to the effective action. Express the final result as a function of ϵ, and extract the divergent contribution at $\epsilon\to 0$.

Hint: Perform the integral over the Feynman parameter at the very end.

Next we introduce the renormalised mass m and coupling λ, and express the bare constants κ, m_0, λ_0 as functions of them

$$\kappa = 1+c_1\lambda^2+\mathcal{O}(\lambda^4),\qquad m_0^2 = m^2\big(1+c_2\lambda^2+\mathcal{O}(\lambda^4)\big),\qquad \lambda_0=\lambda+c_3\lambda^3+\mathcal{O}(\lambda^5). \tag{6.12}$$

e) Write the two-point contributions to the effective action (tree level plus one loop) as a function of the renormalised mass and coupling. Determine the one-loop contributions to (6.12) such that the divergences due to the self energy cancel.

f) Compute the one-loop correction to the vertex. Identify the divergent part, and determine the remaining coefficient in (6.12) to make the effective action finite.

Quantum Field Theory II — Problem Set 7

ETH Zurich, 2026 FS — Prof. N. Beisert

due: 22. 04. 2026

7.1. BCH formula

The Baker–Campbell–Hausdorff (BCH) formula states that

$$\exp(A)\exp(B) = \exp(A + B + A \star B), \tag{7.1}$$

where $A \star B$ is an element of the Lie algebra generated by A and B given by

$$A \star B = \tfrac{1}{2}[\![A, B]\!] + \tfrac{1}{12}[\![A, [\![A, B]\!]]\!] + \tfrac{1}{12}[\![B, [\![B, A]\!]]\!] + \ldots\,. \tag{7.2}$$

Prove the BCH formula to this order assuming that A and B are matrices.

Hint: Replace the exponents X by ϵX and expand both sides to cubic order in ϵ.

7.2. Simple Lie groups and simple Lie algebras

A simple Lie group is a connected non-abelian Lie group with no proper connected normal subgroups (a subgroup H of a group G is called *normal* if $ghg^{-1} \in \mathrm{H}$ for all $h \in \mathrm{H}$, $g \in \mathrm{G}$; a subgroup of a group G is called *proper* if it is neither trivial nor the whole of G). We want to understand what this condition means in terms of the Lie algebra $\mathfrak{g}$ of the group.

In this exercise, we will only consider compact connected Lie groups, for which the exponential map is onto, i.e. each $g \in \mathrm{G}$ can be written as $g = \mathrm{e}^A$ for some $A \in \mathfrak{g}$.

a) A *subalgebra* is a subspace of an algebra which is closed under multiplication. In the case of a Lie algebra this means that a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ obeys

$$\mathfrak{h} \subset \mathfrak{g}, \qquad [\![\mathfrak{h}, \mathfrak{h}]\!] \subset \mathfrak{h}. \tag{7.3}$$

Show that the exponential map maps a subalgebra $\mathfrak{h}$ into a subgroup H of G.

Hint: recall the Baker–Campbell–Hausdorff formula discussed in problem 7.1.

b) An *ideal* $\mathfrak{h}$ of $\mathfrak{g}$ is a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with the property that

$$[\![\mathfrak{g}, \mathfrak{h}]\!] \subset \mathfrak{h}. \tag{7.4}$$

A *proper* subalgebra of $\mathfrak{g}$ is a subalgebra which is neither trivial nor all of $\mathfrak{g}$.

Show, using the exponential map, that a Lie group is simple if and only if its Lie algebra contains no proper ideals.

Hint: it is sufficient to work with infinitesimal elements of the group, i.e. $\exp(\epsilon A) = 1 + \epsilon A + \mathcal{O}(\epsilon^2)$.

$\longrightarrow$

7.3. Dual and complex conjugate representations

The dual of a representation R of a Lie algebra $\mathfrak{g}$ on the vector space $\mathbb{V}$ is defined as the representation R^* on the dual vector space $\mathbb{V}^*$ by $R^*(a) = -R(a)^\mathsf{T}$ for all $a \in \mathfrak{g}$.

a) Verify that R^* is a representation.

b) For a unitary representation R of a real Lie algebra, show that R^* is the complex conjugate representation $\bar{R}$.

c) Show that the tensor product $R \otimes R^*$ contains the trivial representation.
Hint: Consider the representation acting on the state $v^i \otimes v_i^*$, where v^i is a basis of $\mathbb{V}$ and v_i^* is the dual basis of $\mathbb{V}^*$ with $v_i^*(v^k) = \delta_i^k$.

8.1. Tensor product representations

Representations of a Lie algebra $\mathfrak{g}$ can be combined to give bigger representations via direct sums and tensor products

$$R_{1\oplus 2}(a) = \begin{pmatrix} R_1(a) & 0 \\ 0 & R_2(a) \end{pmatrix}, \qquad R_{1\otimes 2}(a) = R_1(a)\otimes \mathrm{id}_2 + \mathrm{id}_1 \otimes R_2(a). \tag{8.1}$$

Here R_1, R_2 are representations of $\mathfrak{g}$ on the spaces $\mathbb{V}_1, \mathbb{V}_2$ and $a \in \mathfrak{g}$.

a) Verify that the tensor product of two representations is again a representation.

b) Consider the tensor product $R^{\otimes 2}$ of two identical representations R. Show that it can be decomposed as the direct sum $R^{\otimes 2} = R^+ \oplus R^-$ of representations $R^\pm$ acting on the symmetric and anti-symmetric subspaces $\mathbb{V}^\pm \subset \mathbb{V}\otimes\mathbb{V}$

$$R^\pm := \mathit{\Pi}^\pm R^{\otimes 2}, \qquad \mathit{\Pi}^\pm := \tfrac{1}{2}(\mathrm{id} \pm \mathrm{ex}), \tag{8.2}$$

where $\mathit{\Pi}^\pm$ are projectors onto the subspaces $\mathbb{V}^\pm$ and ex is the permutation on $\mathbb{V}\otimes\mathbb{V}$ acting as $\mathrm{ex}(v\otimes w) = w \otimes v$ for any two vectors $v, w \in \mathbb{V}$.

Hint: It is useful to show that $\mathit{\Pi}^\pm R^{\otimes 2} = R^{\otimes 2}\mathit{\Pi}^\pm$.

8.2. Killing form and Casimir invariant

The Killing bilinear form for a Lie algebra $\mathfrak{g}$ is defined as

$$K(a,b) := \mathrm{tr}\big(R^{\mathrm{ad}}(a)R^{\mathrm{ad}}(b)\big) \qquad \text{for } a, b \in \mathfrak{g}. \tag{8.3}$$

It can be expanded in a basis T_a of $\mathfrak{g}$ as the matrix $k_{ab} := K(T_a, T_b)$. For semi-simple $\mathfrak{g}$ this matrix is invertible and its inverse shall be denoted by k^{ab}. Define the representation R of the quadratic Casimir invariant $C_2 = K^{-1}$ as

$$R(C_2) := k^{ab} R(T_a) R(T_b). \tag{8.4}$$

a) For generic $a, b, c \in \mathfrak{g}$ show that

$$K(a,b) = K(b,a), \qquad K([\![c,a]\!], b) + K(a, [\![c,b]\!]) = 0. \tag{8.5}$$

Express these relationships in terms of k_{ab} and $f_{abc} := f_{ab}{}^{d} k_{dc}$.

b) Show that $R(C_2)$ is invariant, i.e. $[R(a), R(C_2)] = 0$ for all $a \in \mathfrak{g}$.

c) Argue that $R(C_2) = C_2^R\, \mathrm{id}^R$ for an irreducible representation R.

$\longrightarrow$

8.3. Completeness relation and Casimirs for $\mathfrak{su}(N)$

The special unitary algebra $\mathfrak{su}(N)$ is defined as the commutator algebra on the space of anti-hermitian traceless matrices. We can use an imaginary basis T_a^{def}, $a = 1, \dots N^2 - 1$, so that T_a^{def} is hermitian $(T_a^{\text{def}})^\dagger = T_a^{\text{def}}$. The Killing metric k_{ab} is

$$\operatorname{tr}[T_a^{\text{def}} T_b^{\text{def}}] = B^{\text{def}} k_{ab}. \tag{8.6}$$

a) Let X be a generic $N \times N$ complex matrix. Prove the completeness relation

$$k^{ab} \operatorname{tr}[T_a^{\text{def}} X]\, T_b^{\text{def}} = B^{\text{def}} \big(X - N^{-1} \operatorname{tr}[X]\, \text{id}\big). \tag{8.7}$$

Hint: consider the space of $N \times N$ complex matrices as an N^2-dimensional vector space over $\mathbb{C}$ and find a suitable basis by extending the basis of $\mathfrak{su}(N)$.

b) Knowing the previous identity (8.7), prove the completeness relation

$$k^{ab} T_a^{\text{def}} X T_b^{\text{def}} = B^{\text{def}} \big(\operatorname{tr}[X]\, \text{id} - N^{-1} X\big). \tag{8.8}$$

c) Consider the symmetric structure constants $d_{ab}{}^c$ as defined by

$$\{T_a^{\text{def}}, T_b^{\text{def}}\} = d_{ab}{}^c T_c^{\text{def}} + q\, k_{ab}\, \text{id}\,. \tag{8.9}$$

By choosing the coefficient q appropriately, show that $d_{ab}{}^c$ is traceless in the first two indices,

$$k^{ab} d_{ab}{}^c = 0. \tag{8.10}$$

d) Show that the quadratic and cubic Casimir invariants for the defining representation, $k^{ab} T_a^{\text{def}} T_b^{\text{def}} = C_2^{\text{def}}\, \text{id}$ and $d^{abc} T_a^{\text{def}} T_b^{\text{def}} T_c^{\text{def}} = C_3^{\text{def}}\, \text{id}$, are given by

$$C_2^{\text{def}} = \frac{N^2 - 1}{N}\, B^{\text{def}}, \qquad C_3^{\text{def}} = \frac{(N^2 - 4)(N^2 - 1)}{N^2}\, (B^{\text{def}})^2. \tag{8.11}$$

Quantum Field Theory II — Problem Set 9

ETH Zurich, 2026 FS — Prof. N. Beisert

due: 20. 05. 2026

9.1. Ward–Takahashi identity

The Ward–Takahashi identities in QED are the extension of the conservation of the Noether current for the U(1) global symmetry

$$N_\mu(x) = -\mathring{\imath} q\bar{\psi}(x)\gamma_\mu\psi(x) \tag{9.1}$$

to correlation functions. The Ward–Takahashi identity reads

$$\begin{aligned}&\partial_z^\mu\big\langle N_\mu(z)\,\psi(x_1)\bar{\psi}(y_1)\ldots\psi(x_n)\bar{\psi}(y_n)\,A_{\nu_1}(z_1)\ldots A_{\nu_p}(z_p)\big\rangle\\ &= -q\big\langle\psi(x_1)\bar{\psi}(y_1)\ldots\psi(x_n)\bar{\psi}(y_n)\,A_{\nu_1}(z_1)\ldots A_{\nu_p}(z_p)\big\rangle\sum_{i=1}^{n}\big(\delta^4(z-x_i)-\delta^4(z-y_i)\big).\end{aligned} \tag{9.2}$$

Define the full propagator for the electron in momentum space as

$$-\mathring{\imath}(2\pi)^4 G(k)\delta^4(k-p) := \int \mathrm{d}y^4\,\mathrm{d}z^4\,\big\langle\psi(y)\bar{\psi}(z)\big\rangle\,\mathrm{e}^{\mathring{\imath}p\cdot z - ik\cdot y}\,. \tag{9.3}$$

and the electromagnetic vertex function $V^\mu(k,l)$ as

$$\int \mathrm{d}x^4\,\mathrm{d}y^4\,\mathrm{d}z^4\,\mathrm{e}^{-\mathring{\imath}(p\cdot z+k\cdot x-l\cdot y)}\,\big\langle N^\mu(z)\psi(x)\bar{\psi}(y)\big\rangle =: -\mathring{\imath}(2\pi)^4 qG(k)V^\mu(k,l)G(l)\delta^4(p+k-l). \tag{9.4}$$

a) Show that the Ward–Takahashi identity implies that

$$p_\mu V^\mu(k,k+p) = \mathring{\imath}G^{-1}(p+k) - \mathring{\imath}G^{-1}(k). \tag{9.5}$$

b) We define the counterterm Z_1 as the limit

$$\lim_{p\to 0} V^\mu(k,k+p) = (Z_1)^{-1}\gamma^\mu, \tag{9.6}$$

and we define the counterterm Z_2 as the residue of $G(k)$ near the mass-shell $k^2 \to -m^2$, i.e.

$$G(k) \sim \frac{-Z_2}{\mathring{\imath}\gamma^\mu k_\mu + m} + \cdots \tag{9.7}$$

where the ellipses denote terms which are not singular at $k^2 \to -m^2$.

Note that $V^\mu(k,k)$ is the complete vertex function when the momentum of the external photon goes to zero, and Z_2 is defined for on-shell electrons with momentum k_μ. Show that, with these definitions,

$$Z_1 = Z_2. \tag{9.8}$$

$\longrightarrow$

9.2. Axial anomaly in two dimensions

Consider a massless Dirac spinor field ψ in two spacetime dimensions coupled to a fixed background gauge field A

$$\mathcal{L} = \bar{\psi}\gamma^\mu(\partial_\mu - \mathring{\imath}A_\mu)\psi. \tag{9.9}$$

The Dirac matrices satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. The matrix γ^3 is the two-dimensional analogue of γ^5 in four dimensions

$$\gamma^3 = -\tfrac{1}{2}\varepsilon_{\mu\nu}\gamma^\mu\gamma^\nu = -\gamma^0\gamma^1, \tag{9.10}$$

where $\varepsilon_{\mu\nu}$ is the totally anti-symmetric tensor in two dimensions with $\varepsilon_{01} := +1$. It satisfies

$$\gamma^3\gamma^\mu = -\gamma^\mu\gamma^3 = \eta^{\mu\nu}\varepsilon_{\nu\rho}\gamma^\rho. \tag{9.11}$$

a) Choose the spinor field and gamma-matrices as follows:

$$\psi = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix}0 & +1\\ -1 & 0\end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix}+1 & 0\\ 0 & -1\end{pmatrix}. \tag{9.12}$$

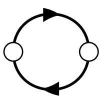
Write the Lagrangian (9.9) in terms of these components, and derive the equations of motion. What do you observe?

b) Use the equations of motion to show that the vector and axial currents

$$N^\mu_{\mathrm{V}} = -\mathring{\imath}\bar{\psi}\gamma^\mu\psi, \qquad N^\mu_{\mathrm{A}} = -\mathring{\imath}\bar{\psi}\gamma^3\gamma^\mu\psi. \tag{9.13}$$

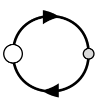
are separately conserved.

c) *optional:* Compute the correlator of two currents at $A = 0$

$$\big\langle N^\mu_{\mathrm{V}}(p)\, N^\nu_{\mathrm{V}}(q)\big\rangle_{A=0} = N^\mu_{\mathrm{V}}(p)$$

$$N^\nu_{\mathrm{V}}(q) = (2\pi)^2\delta^2(p+q)\,\frac{\mathring{\imath}}{\pi}\left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right). \tag{9.14}$$

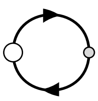
Hint: Use dimensional regularisation and assume massless tadpole integrals to vanish.

d) Express the vector current expectation value $\langle N^\mu_{\mathrm{V}}(p)\rangle$ at one loop and to linear order in the vector field A in terms of the current correlator (9.14). In other words, compute the graph

$$N^\mu_{\mathrm{V}}(p)$$

$$A\,. \tag{9.15}$$

Check that the current is conserved at this order.

e) Show that the axial current is no longer conserved at the quantum level. To this end compute the expectation value $\langle N^\mu_{\mathrm{A}}(p)\rangle$ at one loop and at linear order in A

$$N^\mu_{\mathrm{A}}(p)$$

$$A \tag{9.16}$$

using your results of part d) together with (9.11) to relate the currents.

f) Can you obtain the anomaly from a computation in position space? What do you find for $\langle\partial\cdot N_{\mathrm{V}}(x)\rangle$ and $\langle\partial\cdot N_{\mathrm{A}}(x)\rangle$? Can you make it agree with parts d) and e)?
Hint: No regularisation is required here.

Quantum Field Theory II

ETH Zurich, 2026 FS

Problem Set 10

Prof. N. Beisert

due: 20. 05. 2026

10.1. The Coleman–Weinberg potential

The Coleman–Weinberg model represents quantum electrodynamics of a scalar field in four dimensions. The Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - (D^\mu\phi)^* D_\mu\phi + \mu^2\phi^*\phi - \tfrac{1}{4}\lambda(\phi^*\phi)^2, \tag{10.1}$$

where $\phi(x)$ is a complex scalar field and $D^\mu = \partial^\mu - \mathring{\imath} qA^\mu$ is the covariant derivative.

a) Assume that $\mu^2 > 0$, so that the U(1) symmetry of the Lagrangian is spontaneously broken by means of the Higgs mechanism. Define the complex field ϕ in such a way, that one of its two degrees of freedom is gauged away by a proper gauge transformation and expand the other degree of freedom around its saddle point. Finally by expanding the Lagrangian itself, show that the field A_μ acquires a mass.

b) Consider a generic action $S[\phi]$ of a scalar field ϕ. Show that the one-loop effective action takes the form

$$G[\phi] = S[\phi] + \frac{\mathring{\imath}}{2}\log\operatorname{Det}\left[-\mathring{\imath}\,\frac{\delta^2 S}{\delta\phi^2}\right] + \ldots. \tag{10.2}$$

In order to compute the underlying partition function Z, we employ the so-called background field method and write the quantum scalar field $\hat{\phi}(x)$ as a classical background field $\phi(x)$ plus quantum fluctuations $\eta(x)$

$$\hat{\phi}(x) = \phi(x) + \eta(x). \tag{10.3}$$

Hint: Use the argument ϕ of the effective action $G[\phi]$ as the background field, expand the exponent of $Z[j]$ to quadratic order in η and choose j and ϕ in accordance with the saddle point condition

$$\frac{\delta S}{\delta\phi} + j = 0. \tag{10.4}$$

c) Let us return to the Lagrangian (10.1). Compute the one-loop correction to the effective potential

$$V_{\text{eff}}(\phi) = -G[\phi], \tag{10.5}$$

where ϕ is a constant background field. Eliminate the arising divergences by adjusted minimal subtraction (MS-bar) at the renormalisation scale M.

Hint: After performing the gauge transformation of part a), expand the remaining scalar degree of freedom according to (10.3). Then use (10.2) to compute the one-loop effective action due to the scalar and vector degrees of freedom for which the formula holds equivalently. Note, that only terms quadratic in the fluctuating fields A_μ and η contribute to the functional determinant of (10.2).

d) In the result of part b), take the limit $\mu^2 \to 0$. The result should be an effective potential that is scale-invariant up to logarithms of M. For λ very small, of order q^4, show that $V(\phi)$ has a symmetry-breaking minimum at a value of ϕ for which no logarithm is large and so the perturbation theory analysis is still valid. Thus the $\mu^2 = 0$ theory, for this choice of coupling constants, still has spontaneously broken symmetry, due to the influence of first-order quantum corrections.

⟶

e) The beta-functions for the running couplings q and λ and the gamma-function of ϕ are defined by the equations

$$M\,\frac{\partial\bar{\lambda}}{\partial M} = \beta_\lambda(\bar{\lambda},\bar{q}), \qquad M\,\frac{\partial\bar{q}}{\partial M} = \beta_q(\bar{\lambda},\bar{q}), \qquad M\,\frac{\partial\bar{\phi}}{\partial M} = -\gamma_\phi(\bar{\lambda},\bar{q})\bar{\phi}. \tag{10.6}$$

Explicit calculation yields the following one-loop results

$$\begin{aligned}\beta_q &= \frac{q^3}{48\pi^2}\,,\\ \beta_\lambda &= \frac{15\lambda^2 - 36\lambda q^2 + 72q^4}{32\pi^2}\,,\\ \gamma_\phi &= -\,\frac{2q^2}{16\pi^2}\,.\end{aligned} \tag{10.7}$$

Construct the RG-improved effective potential for the $\mu^2 = 0$ model and discuss qualitatively whether the spontaneous symmetry breaking still occurs.

Finally, compute the mass of the scalar particle m as a function of q^2, λ and M and determine the ratio m/m_A to leading order in q^2, for $\lambda \ll q^2$.

Hint: The RG-improved effective potential should be such that when expanded in terms of the coupling constants around the energy M, it will recover the result in part d).

11.1. The Higgs mechanism in the standard model

In this exercise we review the Higgs mechanism in the standard model. Let us consider first of all the mass terms of matter fields. Take the first lepton generation, i.e. the electron and its neutrino. The electroweak gauge group is $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$ (weak isospin and hypercharge). Left- and right-handed fermions appear in different structures in the Lagrangian. The left-handed leptons form a doublet with respect to $\mathrm{SU}(2)_I$

$$L_\mathrm{L} := \begin{pmatrix} \nu_\mathrm{L} \\ e_\mathrm{L} \end{pmatrix}, \qquad L'_\mathrm{L} = \mathrm{e}^{i\theta^a I_a}\, L_\mathrm{L}, \tag{11.1}$$

where the generators of $\mathrm{SU}(2)_I$ in the fundamental representation may be chosen to be $I_a = \frac{1}{2}\sigma_a$ (σ_a are the Pauli matrices). The right-handed electron is a singlet under $\mathrm{SU}(2)_I$.

$$e'_\mathrm{R} = e_\mathrm{R}. \tag{11.2}$$

Neutrinos are taken to be massless: the corresponding right-handed field is thus not needed.

Under $\mathrm{U}(1)_Y$ the fields transform as

$$L'_\mathrm{L} = \mathrm{e}^{-i\theta/2}\, L_\mathrm{L}, \qquad e'_\mathrm{R} = \mathrm{e}^{-i\theta}\, e_\mathrm{R}. \tag{11.3}$$

a) Why is a mass term of the form

$$\mathcal{L}_\mathrm{mass} := -m_e(\bar{e}_\mathrm{L} e_\mathrm{R} + \mathrm{h.c.}), \tag{11.4}$$

not allowed?

b) In order to assign a mass to our fields we introduce a new scalar field H. This field couples to the electron and neutrino as

$$\mathcal{L}_\mathrm{Yukawa} := -y(\bar{L}_\mathrm{L} H e_\mathrm{R} + \mathrm{h.c.}), \tag{11.5}$$

where y is the coupling constant. How should H change with respect to a gauge transformation?

c) Show that it is always possible to find a gauge transformation U that sets the first component of H to zero and makes the second one real,

$$UH = U\begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} 0 \\ H_\mathrm{r} \end{pmatrix}. \tag{11.6}$$

The gauge that brings H in this form is called *unitary gauge*.

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d) Consider a gauge-invariant, renormalisable potential term for the scalar field

$$\mathcal{L}_{\text{pot}} = \mu^2(H^\dagger H) - \tfrac{1}{2}\lambda(H^\dagger H)^2. \tag{11.7}$$

Find the minima of this potential and show that for $\mu^2 > 0$ they correspond to field configuration(s) $H \neq 0$.

e) Assume $\mu^2 > 0$ and redefine

$$H_{\text{r}} = \frac{1}{\sqrt{2}}\,(v+\eta), \tag{11.8}$$

to make excitations of H with respect to the ground state more explicit. η is thus the new physical Higgs boson field. Write down (11.5) in terms of the Higgs field and the vacuum expectation value v. Note that the fermions acquire a mass term, and express the fermion mass in terms of y and v. What is the coupling of the Higgs boson to the fermions in terms of m_e and v?

The standard model Lagrangian density also contains a kinetic term for the Higgs doublet

$$\mathcal{L}_{\text{kin}} := -(D_\mu H)^\dagger(D^\mu H). \tag{11.9}$$

The covariant derivative acting on the Higgs doublet involves the gauge fields for the $\mathrm{SU}(2)_I$ and $\mathrm{U}(1)_Y$ symmetry W^a_μ and B_μ, respectively

$$D_\mu = \partial_\mu - \mathring{\imath} g I_a W^a_\mu - \mathring{\imath} g' Y B_\mu. \tag{11.10}$$

In the following you should try to find the masses and the couplings to the Higgs boson of the physical standard model fields.

f) Diagonalise the quadratic term by introducing the physical fields

$$W^+_\mu = (W^-_\mu)^\dagger = \frac{1}{\sqrt{2}}\,(W^1_\mu - \mathring{\imath} W^2_\mu), \qquad Z^0_\mu = \frac{gW^3_\mu - g'B_\mu}{\sqrt{g^2+g'^2}}\,, \qquad A_\mu = \frac{gB_\mu + g'W^3_\mu}{\sqrt{g^2+g'^2}}\,. \tag{11.11}$$

Read off the masses and couplings of the theory in terms of g, g', v, λ and μ by comparing the resulting expression for $\mathcal{L}_{\text{kin}}$ to

$$\begin{aligned}(D_\mu H)^\dagger(D^\mu H) = {} & \tfrac{1}{2}(\partial_\mu\eta)^2 \\ & + \tfrac{1}{2}m_Z^2 Z^0_\mu Z^{0,\mu} + \tfrac{1}{2}\rho_Z \eta Z^0_\mu Z^{0,\mu} + \tfrac{1}{4}\lambda_Z \eta^2 Z^0_\mu Z^{0,\mu} \\ & + m_W^2 W^+_\mu W^{-,\mu} + \rho_W \eta W^+_\mu W^{-,\mu} + \tfrac{1}{2}\lambda_W \eta^2 W^+_\mu W^{-,\mu}.\end{aligned} \tag{11.12}$$

g) The original doublet H included four degrees of freedom. Which symmetry was broken? Where are the degrees of freedom hidden after spontaneous symmetry breaking? Why did the photon field A_μ not acquire a mass term? Why does it couple to the fermion vector currents, but not to the axial currents?