Quantum Field Theory I

Problem Sets

ETH Zurich, 2025 HS

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The current version of this work can be found at:
http://people.phys.ethz.ch/~nbeisert/lectures/.

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Quantum Field Theory I — Problem Set 1

ETH Zurich, 2025 HS — Prof. N. Beisert

due: 23. 09. 2025

1.1. Classical particle in an electromagnetic field

Consider the classical Lagrangian density of a particle of mass m and charge q, moving in an electromagnetic field, specified by the electric potential $\phi(\vec{x},t)$ and the magnetic vector potential $\vec{A}(\vec{x},t)$:

$$\mathcal{L} = \tfrac{1}{2}m\dot{\vec{x}}^2 + q\vec{A}\cdot\dot{\vec{x}} - q\phi. \tag{1.1}$$

Determine the following quantities, and compare the results to those for a free particle:

a) the canonical momentum p_i conjugate to the coordinate x^i;

b) the equations of motion corresponding to the Lagrangian density;

c) the Hamiltonian density of the system.

1.2. Relativistic point particle

The action of a relativistic point particle is given by

$$S = -\alpha \int_{\mathcal{P}} \mathrm{d}\tau \tag{1.2}$$

with the proper time (relativistic time-like line element)

$$\mathrm{d}\tau^2 = -\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \tag{1.3}$$

and α a (yet to be determined) constant.

The path $\mathcal{P}$ between two points x_1^μ and x_2^μ can be parametrised by a parameter λ. With that, the integral of the line element $\mathrm{d}\tau$ becomes an integral over the parameter

$$S = -\alpha \int_{\tau_1}^{\tau_2} \mathrm{d}\lambda \sqrt{-\eta_{\mu\nu}\frac{\partial x^\mu}{\partial\lambda}\frac{\partial x^\nu}{\partial\lambda}}\,. \tag{1.4}$$

a) Parametrise the path by the time coordinate $\lambda = t = x^0$ and take the non-relativistic limit $\|\dot{\vec{x}}\| \ll 1$ to determine the value of the constant α.

b) Derive the equations of motion by varying the action.

Hint: You may want to determine the canonically conjugate momentum first.

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1.3. Coherent quantum oscillator

Consider the Hamiltonian of a quantum harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} . \tag{1.5}$$

a) Introduce ladder operators to diagonalise the Hamiltonian.

b) Calculate the expectation values of the number operator $N \sim a^\dagger a$ as well as of the x and p operator in a general number state $|n\rangle$.

c) Calculate the variances $\varDelta x$, $\varDelta p$ and $\varDelta N$ in the same state $|n\rangle$ and use them to determine the Heisenberg uncertainty of $|n\rangle$.

d) Show that the coherent state

$$|\alpha\rangle = \mathrm{e}^{\alpha p} |0\rangle \tag{1.6}$$

is an eigenstate of the annihilation operator you defined in part a).

e) Calculate the time-dependent expectation values of x, p and N,

$$\langle\alpha|x(t)|\alpha\rangle, \qquad \langle\alpha|p(t)|\alpha\rangle, \qquad \langle\alpha|N(t)|\alpha\rangle, \tag{1.7}$$

as well as the corresponding variances to determine the uncertainty of the state $|\alpha\rangle$. Compare your result with the result obtained in part c).

1.4. Coupled pendula

We consider a system of three identical pendula of length ℓ and mass m in a homogeneous gravitational field with acceleration g. The pendula are moving in the same plane and we denote the (small) deflection angles by θ_j, $j = 1, 2, 3$. Moreover, the pendula are connected by massless springs whose length equals the distance of the pendula in their gravitational ground state $\theta_j = 0$. At first, consider two springs of equal spring constant k connecting the pairs of pendula $1, 2$ and $2, 3$.

a) Find the equations of motion for θ_k and the normal modes of the system.

Now add a third spring of spring constant k' connecting pendula $1, 3$.

b) Find the equations of motion for θ_k and the normal modes of the system.

c) *optional:* For some value of k' the system behaves in a special way. Can you determine k'? Can you tell in what sense the system becomes special?

Quantum Field Theory I — Problem Set 2

ETH Zurich, 2025 HS — Prof. N. Beisert

due: 30. 09. 2025

2.1. Integral definition of the step function

In this exercise we will demonstrate that:

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\theta(x) = \delta(x), \tag{2.1}$$

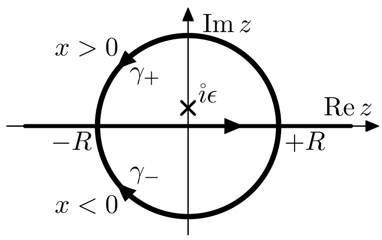
where:

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases} \qquad \int_A \mathrm{d}x\,\delta(x)\,f(x) = \begin{cases} f(0) & \text{if } 0 \in A, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

To that end, define the function F via the integral (where $x \in \mathbb{R}$, $\epsilon > 0$):

$$F(x,\epsilon) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \mathrm{d}z\,\frac{\mathrm{e}^{ixz}}{z - i\epsilon}\,. \tag{2.3}$$

a) Consider a semi-circular path $\gamma_\pm(R)$ of radius R in the upper/lower half of the complex plane and ending on the real axis.



(2.4)

Argue that:

$$\begin{aligned} \lim_{R\to\infty} \int_{\gamma_+(R)} \mathrm{d}z\,\frac{\mathrm{e}^{ixz}}{z - i\epsilon} &= 0 \qquad \text{if } x > 0, \\ \lim_{R\to\infty} \int_{\gamma_-(R)} \mathrm{d}z\,\frac{\mathrm{e}^{ixz}}{z - i\epsilon} &= 0 \qquad \text{if } x < 0. \end{aligned} \tag{2.5}$$

Hint: Use integration by parts to improve convergence of the integral.

b) Consider the closed paths $\Gamma_\pm(R) := [-R,+R] \cup \gamma_\pm(R)$ and make use of the Cauchy integral formula

$$\frac{1}{2\pi i}\oint_\Gamma \mathrm{d}z\,f(z) = \operatorname*{res}_\Gamma f, \tag{2.6}$$

where $\mathrm{res}_\Gamma f$ is the sum of the residues of the poles of f surrounded by the contour Γ, to argue that:

$$F(x,\epsilon) = \theta(x)\,\mathrm{e}^{-\epsilon x}\,. \tag{2.7}$$

c) Finally, using (2.3) and (2.7), argue that relation (2.1) holds. You will have to take some mathematically questionable steps. Which are they precisely? Can they be justified? How?

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2.2. Discrete and continuous treatment of a 1D spring lattice

Consider a one-dimensional array of N particles at positions $q_i(t)$, $i = 1, \ldots, N$ connected by elastic springs with spring force constant κ. Assume that all particles have mass μ, and their relative distance at rest is a.

a) Derive the Lagrangian $L(q_i(t), \dot{q}_i(t))$ of this system and compute the Euler–Lagrange equations.

b) Determine the continuum form of these equations by taking the simultaneous limit $a \to 0$ and $N \to \infty$: Approximate the discrete displacements by a smooth field $\phi(x,t)$ as $q_i(t) = ia + \phi(ia, t)$ where ia is the rest position on the spring lattice. In taking the limit, assume the resulting length $Na \to L$, the mass density $\mu/a \to \bar{\mu}$ and the elastic modulus $\kappa a \to \bar{\kappa}$ to be finite.

c) Directly take the continuum limit of $L(q_i(t), \dot{q}_i(t))$ and show that the Euler–Lagrange equations for the Lagrangian density $\mathcal{L}(\phi, \phi', \dot{\phi})$ are the same as those obtained in part b).

2.3. Normal modes of a Klein–Gordon field

Consider a real scalar field $\phi(x,t)$ in one spatial dimension with the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}\phi'^2 - \tfrac{1}{2}m^2\phi^2, \tag{2.8}$$

on the interval $x \in [0, L]$ with the boundary conditions $\phi(0,t) = \phi(L,t) = 0$. In this exercise we consider the normal modes for this field.

a) Derive the Euler–Lagrange equations. Assuming that solutions factorise as $\phi(x,t) = \psi(x)\theta(t)$, show that the equations of motion reduce to the following pair of separated differential equations:

$$\psi''(x) + k^2\psi(x) = 0, \qquad \ddot{\theta}(t) + \omega^2\theta(t) = 0, \tag{2.9}$$

for some $k, \omega \in \mathbb{R}$. Express $\omega = \omega(k)$ in terms of k.

b) Determine the allowed values k_n for the momentum k. For each $n \in \mathbb{Z}_+$ find the corresponding spatial solution $\psi_n(x)$ normalised by:

$$\int_0^L \mathrm{d}x\, \psi_n(x)\psi_{n'}(x) = \delta_{nn'}. \tag{2.10}$$

c) Show that the general solution can be written as:

$$\phi(x,t) = \sum_{n=1}^{\infty} \big[a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\big]\psi_n(x). \tag{2.11}$$

Express the constants a_n and b_n in terms of the initial conditions $\phi(x,0) = \phi(x)$ and $\dot{\phi}(x,0) = \dot{\phi}(x)$, where $\phi(x)$ and $\dot{\phi}(x)$ denote a pair of independent functions of space.

d) Show that the non-trivial Poisson brackets take the form $\{a_n, b_{n'}\} = \delta_{nn'}/\omega_n$.

e) Show that the total energy of the general solution (2.11) in terms of a_n, b_n can be written as:

$$E[\phi] = \int_0^L \mathrm{d}x\, \big[\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\phi'^2 + \tfrac{1}{2}m^2\phi^2\big] = \sum_{n=1}^{\infty} \tfrac{1}{2}\omega_n^2\big(a_n^2 + b_n^2\big). \tag{2.12}$$

Quantum Field Theory I — Problem Set 3

ETH Zurich, 2025 HS — Prof. N. Beisert

due: 07. 10. 2025

3.1. Scalar field correlator

In this problem we shall consider the amplitude for a particle to be created at point y and annihilated at point x

$$C(y,x) := \mathring{\imath}\langle 0|\phi(y)\phi(x)|0\rangle. \tag{3.1}$$

a) Use the Fourier expansion of $\phi(x)$ to show the following integral expression for $C(y,x)$ with $p^0 = e(\vec{p}) := \sqrt{\vec{p}^2 + m^2}$

$$C(y,x) = \mathring{\imath} \int \frac{\mathrm{d}\vec{p}^3}{(2\pi)^3\, 2e(\vec{p})}\, \mathrm{e}^{\mathring{\imath} p\cdot(y-x)}\,. \tag{3.2}$$

b) Observe that the amplitude satisfies

$$C(y,x) = C(y-x,0) =: C(y-x). \tag{3.3}$$

What are the properties of C under translations and Lorentz transformations?

c) Use Cauchy's residue theorem to show that $C(x)$ can be also written as

$$C(x) = -\int_\gamma \frac{\mathrm{d}p^4}{(2\pi)^4}\, \frac{\mathrm{e}^{\mathring{\imath} p\cdot x}}{p^2+m^2}\,, \tag{3.4}$$

where the integration over the contour γ given in the left figure of (3.7) corresponds to the (complex) variable p^0.

d) Show that $C(x)$ satisfies the Klein–Gordon equation, i.e.

$$(-\partial^2 + m^2)C(x) = 0. \tag{3.5}$$

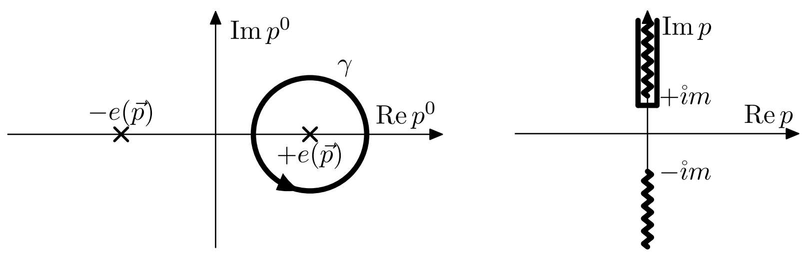
e) Express $C(x)$ for a time-like x as a single integral over the energy, and the one for space-like x as a single integral over $p = \|\vec{p}\|$.

Hint: Use a Lorentz transformation to reduce to the cases $\vec{x} = 0$ and $x^0 = 0$ respectively.

f) For space-like x with $r := \|\vec{x}\|$, show the asymptotic behaviour of the correlator

$$C(x) \sim \mathrm{e}^{-mr} \quad \text{for } r \to \infty. \tag{3.6}$$

Hint: Wrap the contour of integration around the upper branch cut of the integrand as depicted in the right figure of (3.7). Use the fact that there is a phase shift of $\mathrm{e}^{\mathring{\imath}\pi} = -1$ between the two sides of the branch cut.



(3.7)

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3.2. Commutator and causality

In order to know whether a measurement of the field at x can affect another measurement at y, one may compute the commutator

$$\Delta(y-x) := \mathring{\imath}\langle 0|\big[\phi(y),\phi(x)\big]|0\rangle = C(y-x) - C(x-y). \tag{3.8}$$

Show that such a commutator vanishes for a space-like separation of x and y, which proves that causality is obeyed.

3.3. Complex scalar field

We want to investigate the theory of a complex scalar field $\phi = \phi(x)$. The theory is described by the Lagrangian density:

$$\mathcal{L} = -\partial_\mu\phi^\dagger\,\partial^\mu\phi - m^2\phi^\dagger\phi. \tag{3.9}$$

As a complex scalar field has two degrees of freedom, we can treat ϕ and $\phi^\dagger$ as independent fields with one degree of freedom each.

a) Find the conjugate momenta $\pi(\vec{x})$ and $\pi^\dagger(\vec{x})$ to $\phi(\vec{x})$ and $\phi^\dagger(\vec{x})$ and the canonical commutation relations. *Note:* we choose $\pi = \partial\mathcal{L}/\partial\dot{\phi}^\dagger$ rather than $\pi = \partial\mathcal{L}/\partial\dot{\phi}$.

b) Find the Hamiltonian of the theory.

c) Introduce creation and annihilation operators to diagonalise the Hamiltonian.

d) Show that the theory contains two sets of particles of mass m.

e) Consider the conserved charge

$$Q = \mathring{\imath}\int \mathrm{d}\vec{x}^3\,(\pi^\dagger\phi - \phi^\dagger\pi). \tag{3.10}$$

Rewrite it in terms of ladder operators and determine the charges of the two particle species.

3.4. Number and momentum operators

a) Consider a real scalar field with creation and annihilation operators $a^\dagger(\vec{p}), a(\vec{p})$. The number and momentum operators are defined as

$$N = \int \frac{\mathrm{d}\vec{p}^d}{(2\pi)^d\,2e(\vec{p})}\,a^\dagger(\vec{p})a(\vec{p}), \qquad \vec{P} = \int \frac{\mathrm{d}\vec{p}^d}{(2\pi)^d\,2e(\vec{p})}\,\vec{p}\,a^\dagger(\vec{p})a(\vec{p}). \tag{3.11}$$

A generic two-particle state is given by

$$|\psi\rangle = \int \frac{\mathrm{d}\vec{p}^d}{(2\pi)^d\,2e(\vec{p})}\,\frac{\mathrm{d}\vec{q}^d}{(2\pi)^d\,2e(\vec{q})}\,f(\vec{p},\vec{q})\,a^\dagger(\vec{p})a^\dagger(\vec{q})|0\rangle \tag{3.12}$$

with a symmetric wave function $f(\vec{p},\vec{q}) = f(\vec{q},\vec{p})$.

Show explicitly that the state $|\psi\rangle$ is an eigenstate of N. Compute $\vec{P}|\psi\rangle$.

b) Now consider a complex scalar field with particle operators $a(\vec{p}), a^\dagger(\vec{p})$ and antiparticle operators $b(\vec{p}), b^\dagger(\vec{p})$. A particle-antiparticle state is

$$|\psi\rangle = \int \frac{\mathrm{d}\vec{p}^d}{(2\pi)^d\,2e(\vec{p})}\,\frac{\mathrm{d}\vec{q}^d}{(2\pi)^d\,2e(\vec{q})}\,f(\vec{p},\vec{q})\,a^\dagger(\vec{p})b^\dagger(\vec{q})|0\rangle \tag{3.13}$$

with an unconstrained wave function $f(\vec{p},\vec{q})$.

How do you write N and $\vec{P}$ in this case? What changes in the results? Explain.

Quantum Field Theory I — Problem Set 4

ETH Zurich, 2025 HS — Prof. N. Beisert

due: 14. 10. 2025

4.1. Retarded propagator

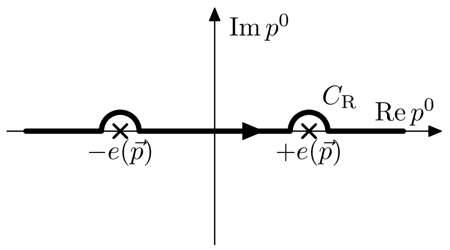
Consider the commutator $\Delta(y-x) = \mathring{\imath}[\phi(y), \phi(x)]$ and define

$$G_{\mathrm{R}}(y-x) := \theta(y^0 - x^0)\, \Delta(y-x), \tag{4.1}$$

which clearly vanishes for any $y^0 < x^0$.

a) Show that

$$G_{\mathrm{R}}(x) = \int_{C_{\mathrm{R}}} \frac{\mathrm{d}p^4}{(2\pi)^4} \frac{\mathrm{e}^{\mathring{\imath} p\cdot x}}{p^2 + m^2}\,, \tag{4.2}$$



with the contour C_{R} given in the figure.

b) Check that $G_{\mathrm{R}}(x)$ is a Green function for the Klein–Gordon equation,

$$(-\partial^2 + m^2) G_{\mathrm{R}}(x) = \delta^4(x). \tag{4.3}$$

4.2. Conservation of charge with complex scalar fields

Consider a free complex scalar field described by

$$\mathcal{L} = -(\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi. \tag{4.4}$$

a) Show that the transformation

$$\phi(x) \to \phi'(x) = \mathrm{e}^{\mathring{\imath}\alpha}\, \phi(x) \tag{4.5}$$

leaves the Lagrangian density invariant.

b) Find the conserved current associated with this symmetry.

If we now consider two complex scalar fields, the Lagrangian density is given by

$$\mathcal{L} = -(\partial_\mu \phi^*_a)(\partial^\mu \phi^a) - m^2 \phi^*_a \phi^a, \quad a = 1, 2. \tag{4.6}$$

c) Show that

$$\phi^a(x) \to \phi'^{\,a}(x) = U^a{}_b \phi^b(x) \tag{4.7}$$

with $U \in \mathrm{U}(2) = \{A \in \mathbb{C}^{2\times 2}; A^{-1} = A^\dagger = (A^*)^{\mathsf{T}}\}$ is a symmetry transformation.

d) Show that now there are four conserved charges: one given by the generalisation of part b), and the other three given by

$$Q_i = -\frac{\mathring{\imath}}{2} \int \mathrm{d}\vec{x}^3 \left(\phi^*_a (\sigma^i)^a{}_b \pi^b - \pi^*_a (\sigma^i)^a{}_b \phi^b\right), \tag{4.8}$$

where σ^i are the Pauli matrices.

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4.3. Symmetry of the stress-energy tensor

Consider a relativistic scalar field theory specified by some Lagrangian $\mathcal{L}(\phi, \partial\phi)$.

a) Compute the variation of $\mathcal{L}(\phi(x), \partial\phi(x))$ under infinitesimal Lorentz transformations (note: $\omega^{\mu\nu} = -\omega^{\nu\mu}$)

$$x^\mu \to x^\mu - \omega^\mu{}_\nu x^\nu + \ldots. \tag{4.9}$$

b) Assuming that $\mathcal{L}(x)$ transforms as a scalar field, i.e. just like $\phi(x)$, derive another expression for its variation under Lorentz transformations.

c) Compare the two expressions to show that the two indices of the stress-energy tensor are symmetric

$$T^{\mu\nu} = -\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\partial^\nu\phi + \eta^{\mu\nu}\mathcal{L} = T^{\nu\mu}. \tag{4.10}$$

4.4. Representations of $\mathfrak{su}(2)$

The Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ is a three-dimensional vector space with basis $\{T_1, T_2, T_3\}$ equipped with an anti-symmetric bilinear Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the Jacobi identity. It is defined by $[T_a, T_b] = \mathring{\imath}\varepsilon_{abc}T_c$, with ε_{abc} the totally anti-symmetric tensor. We will study some of its basic representations.

a) The quadratic Casimir element is defined as $C_2 := \sum_{c=1}^{3} T_c T_c$. Show that $[C_2, T_a] = 0$ for all $a = 1, 2, 3$.

A representation R of $\mathfrak{g}$ is a Lie algebra homomorphism of $\mathfrak{g}$ into $\mathrm{End}(\mathbb{V})$ for some vector space $\mathbb{V}$, i.e. it satisfies

$$R\big([x, y]\big) = \big[R(x), R(y)\big] \qquad \text{for all } x, y \in \mathfrak{g}. \tag{4.11}$$

b) Verify that $R_{1/2}(T_a) = \frac{1}{2}\sigma_a$ with σ_a the Pauli matrices describes a two-dimensional representation of $\mathfrak{su}(2)$, and compute $R_{1/2}(C_2)$.

c) A three-dimensional representation of $\mathfrak{su}(2)$ is given by R_1:

$$R_1(T_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathring{\imath} \\ 0 & +\mathring{\imath} & 0 \end{pmatrix}, \quad R_1(T_2) = \begin{pmatrix} 0 & 0 & +\mathring{\imath} \\ 0 & 0 & 0 \\ -\mathring{\imath} & 0 & 0 \end{pmatrix}, \quad R_1(T_3) = \begin{pmatrix} 0 & -\mathring{\imath} & 0 \\ +\mathring{\imath} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.12}$$

Again, confirm that it is a representation of $\mathfrak{su}(2)$, and compute $R_1(C_2)$. What can you say about $R_1(C_2)$ compared to $R_{1/2}(C_2)$?

Elements of the associated group SU(2) are obtained by matrix exponentiation, that is

$$G(\vec{\theta}) = \exp(\mathring{\imath}\vec{\theta}\cdot\vec{T}) \in \mathrm{SU}(2). \tag{4.13}$$

This transformation describes a finite rotation by the angle $\|\theta\|$ about the axis $\vec{\theta}/\|\theta\|$.

d) Consider a rotation by an angle φ around the z-axis ($\vec{\theta} = \varphi\vec{e}_3$) for both representations. What do you notice? Can you attribute a physical interpretation to your observation?

5.1. Representations of the Lorentz algebra

The Lie algebra $\mathfrak{so}(d,1)$ of the Lorentz group $\mathrm{SO}(d,1)$ for $(d+1)$-dimensional spacetime is given in terms of its generators $M^{\mu\nu}$ (the relativistic angular momentum tensor),

$$[M^{\mu\nu}, M^{\lambda\kappa}] = \mathring{\imath}\big(\eta^{\mu\kappa} M^{\nu\lambda} + \eta^{\nu\lambda} M^{\mu\kappa} - \eta^{\nu\kappa} M^{\mu\lambda} - \eta^{\mu\lambda} M^{\nu\kappa}\big). \tag{5.1}$$

Any representation of the Lorentz algebra must satisfy the above commutation relation.

a) Show explicitly that the following generators $J^{\mu\nu}$ of the vector representation satisfy the Lie algebra

$$(J^{\mu\nu})^{\rho}{}_{\sigma} := \mathring{\imath}(\eta^{\mu\rho}\,\delta^{\nu}_{\sigma} - \eta^{\nu\rho}\,\delta^{\mu}_{\sigma}). \tag{5.2}$$

b) Show explicitly that the following differential operators $L^{\mu\nu}$ satisfy the Lie algebra

$$L^{\mu\nu} := \mathring{\imath}(x^{\mu}\,\partial^{\nu} - x^{\nu}\,\partial^{\mu}). \tag{5.3}$$

5.2. Classical field momentum

Consider the Lagrangian of a real scalar field $\phi = \phi(x)$:

$$\mathcal{L} = -\tfrac{1}{2}\partial_{\mu}\phi\,\partial^{\mu}\phi - \tfrac{1}{2}m^2\phi^2. \tag{5.4}$$

a) Show that the Noether charge corresponding to spatial translations is given by:

$$P^i = -\int \mathrm{d}\vec{x}^3\,\pi\,\partial^i\phi. \tag{5.5}$$

b) Show that the momentum reduces to the following form using Fourier modes:

$$\vec{P} = \int \frac{\mathrm{d}\vec{p}^3}{(2\pi)^3\,2e(\vec{p})}\,\vec{p}\,a^*(\vec{p})a(\vec{p}). \tag{5.6}$$

c) Calculate the Poisson bracket $\{P^i, \phi(\vec{x})\}$ and interpret the result.

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5.3. Properties of gamma-matrices

The gamma-matrices in D dimensional spacetime satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,\mathrm{id}\,. \tag{5.7}$$

Derive the following identities using this algebraic relation (rather than an explicit matrix representation).

a) Prove the following contraction identities

$$\begin{aligned} \gamma^\mu\gamma_\mu &= D\,\mathrm{id}\,, \\ \gamma^\mu\gamma^\nu\gamma_\mu &= -(D-2)\gamma^\nu, \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= (D-4)\gamma^\nu\gamma^\rho + 4\eta^{\nu\rho}\,\mathrm{id}\,, \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -(D-4)\gamma^\nu\gamma^\rho\gamma^\sigma - 2\gamma^\sigma\gamma^\rho\gamma^\nu. \end{aligned} \tag{5.8}$$

b) Show that a trace of an odd number n of gamma-matrices is zero for an even number of spacetime dimensions D

$$\mathrm{tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) = 0. \tag{5.9}$$

Hint: Eliminate double indices, insert $\mathrm{id} = (\gamma^\rho)^{-1}\gamma^\rho$ for some index value ρ (no summation convention implied), and use cyclicity of the trace.

c) Show the following trace identities

$$\begin{aligned} \mathrm{tr}(\gamma^\mu\gamma^\nu) &= \mathrm{tr}(\mathrm{id})\,\eta^{\mu\nu}, \\ \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= \mathrm{tr}(\mathrm{id})\,(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}). \end{aligned} \tag{5.10}$$

6.1. Dirac and Weyl representations of the gamma-matrices

Using the Pauli matrices σ^i, $i = 1, 2, 3$, together with the 2×2 identity matrix σ^0,

$$\sigma^0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.1}$$

we can realise the Dirac representation of the gamma-matrices,

$$\gamma^0_{\mathrm{D}} := i\sigma^3 \otimes \sigma^0, \qquad \gamma^j_{\mathrm{D}} := \sigma^1 \otimes \sigma^j \quad (j = 1, 2, 3), \tag{6.2}$$

where the tensor product can be written as a 4×4 matrix in 2×2 block form as follows

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{pmatrix}. \tag{6.3}$$

We denote the Pauli matrices collectively by $\sigma^\mu = (\sigma^0, \vec{\sigma})$ and define $\bar{\sigma}^\mu = (-\sigma^0, \vec{\sigma})$. We can then define the gamma-matrices in the Weyl representation

$$\gamma^\mu_{\mathrm{W}} := \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{6.4}$$

Show that both representations satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,\mathrm{id}$.
Can you show their equivalence, i.e. $\gamma^\mu_{\mathrm{W}} = T\gamma^\mu_{\mathrm{D}}T^{-1}$ for some matrix T?
Hint: It may help to express the γ^μ_{W} as tensor products of Pauli matrices.

6.2. Spinor rotations

The Dirac equation is invariant under Lorentz transformations $\Psi'(x') = S\Psi(x)$ if the spinor transformation matrix S satisfies

$$\Lambda^\mu{}_\nu\, S^{-1}\gamma^\nu S = \gamma^\mu. \tag{6.5}$$

For an infinitesimal Lorentz transformation $\Lambda_{\mu\nu} = \eta_{\mu\nu} + \delta\omega_{\mu\nu} + \ldots$ this is fulfilled if

$$S = 1 - \tfrac{1}{4}\delta\omega_{\mu\nu}\gamma^\mu\gamma^\nu + \ldots = 1 - \tfrac{1}{8}\delta\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu] + \ldots. \tag{6.6}$$

a) Find the infinitesimal spinor transformation δS for a rotation around the z-axis, i.e. the only non-zero components of $\delta\omega_{\mu\nu}$ are $\delta\omega_{12} = -\delta\omega_{21} \neq 0$.

b) Finite transformations are obtained by exponentiation,

$$S = \exp\bigl(-\tfrac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu\bigr) = \exp\bigl(-\tfrac{1}{8}\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu]\bigr). \tag{6.7}$$

Compute the finite rotation with angle ω_{12} around the same axis as before. Also compute the finite transformation $\Lambda = \exp(\omega)$ for vectors.

c) What happens to the individual components of a spinor under this transformation? What is the period of the transformation in the angle ω_{12}? Compare it to the finite rotation for vectors.

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6.3. Completeness for gamma-matrices

An arbitrary product of gamma-matrices for $D = 4$ is proportional to one of the following 16 linearly independent matrices Γ^a (here a is a multi-index which specifies the type of matrix, S, P, V, A, T, along with the corresponding spacetime indices if any)

$$\begin{aligned} &\Gamma^{\mathrm{S}} := 1, &&\Gamma^{\mathrm{V},\mu} := \gamma^\mu, &&\Gamma^{\mathrm{T},\mu\nu} = -\Gamma^{\mathrm{T},\nu\mu} := \tfrac{\mathring{\imath}}{2}[\gamma^\mu, \gamma^\nu], \\ &\Gamma^{\mathrm{P}} := \gamma^5, &&\Gamma^{\mathrm{A},\mu} := \mathring{\imath}\gamma^5\gamma^\mu. \end{aligned} \tag{6.8}$$

a) Argue that any product of gamma-matrices can indeed be written as a linear combination of the above.

b) Show that $\mathrm{tr}(\Gamma^a) = 4\delta^{a\mathrm{S}}$.

c) Show that the trace of any product of Γ's is given by $\mathrm{tr}(\Gamma^a\Gamma^b) = 4\eta^{ab}$. Here, $\eta^{ab} = 0$ if a and b are of different type, otherwise $\eta^{\mathrm{SS}} = \eta^{\mathrm{PP}} := 1$, $\eta^{(\mathrm{V},\mu)(\mathrm{V},\nu)} = \eta^{(\mathrm{A},\mu)(\mathrm{A},\nu)} := \eta^{\mu\nu}$ and $\eta^{(\mathrm{T},\mu\nu)(\mathrm{T},\rho\sigma)} := \eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}$.

d) Argue that for any pair of multi-indices a, b there is a multi-index c and a factor $\alpha \in \mathbb{C}$ such that $\Gamma^a\Gamma^b = \alpha\Gamma^c$. Make a table of which type of Γ^c you expect for the different types $a, b = \mathrm{S}, \mathrm{P}, \mathrm{V}, \mathrm{A}, \mathrm{T}$. For which products will the result be proportional to Γ^{S}?

e) Show that the 16 matrices are linearly independent and therefore form a complete basis of 4×4 spinor matrices. *Hint:* To do this consider a sum $\sum_a \beta_a\Gamma^a = 0$. What can be said about the coefficients?

6.4. Fierz identity

a) Use the linear independence of the matrices Γ^a to show that

$$\delta^\alpha_\gamma\delta^\beta_\delta = \sum_{a,b} \tfrac{1}{4}\eta_{ab}(\Gamma^a)^\alpha{}_\delta(\Gamma^b)^\beta{}_\gamma, \tag{6.9}$$

where η_{ab} is the inverse of η^{ab} defined in problem 6.3c).
Hint: Decompose an arbitrary matrix $M = \sum_a m_a\Gamma^a$, find the coefficients m_a, substitute the solution back into the original relation, and expand for the elements $M^\alpha{}_\beta$. Alternatively, you can contract the relationship with $(\Gamma^c)^\gamma{}_\beta$.

b) Show that the above relationship can be expressed in components as

$$\begin{aligned} \delta^\alpha_\gamma\delta^\beta_\delta = {}& \tfrac{1}{4}\delta^\alpha_\delta\delta^\beta_\gamma + \tfrac{1}{4}(\gamma^5)^\alpha{}_\delta(\gamma^5)^\beta{}_\gamma + \tfrac{1}{4}(\gamma^\mu)^\alpha{}_\delta(\gamma_\mu)^\beta{}_\gamma \\ & - \tfrac{1}{4}(\gamma^5\gamma^\mu)^\alpha{}_\delta(\gamma^5\gamma_\mu)^\beta{}_\gamma - \tfrac{1}{32}[\gamma^\mu, \gamma^\nu]^\alpha{}_\delta[\gamma_\mu, \gamma_\nu]^\beta{}_\gamma. \end{aligned} \tag{6.10}$$

Hint: It makes sense to contract an anti-symmetric pair of indices with a conventional symmetry factor of $1/2$, e.g. $\sum_a m_a\Gamma^a = \ldots + \frac{1}{2}m_{\mathrm{T},\mu\nu}\Gamma^{\mathrm{T},\mu\nu}$, in order to account for the fact that every summand appears twice in the contraction.

c) Use the result from part a) to show the Fierz identity:

$$X^\alpha{}_\beta Y^\gamma{}_\delta = \sum_{c,d,e,f} \tfrac{1}{16}\eta_{ce}\eta_{df}\,\mathrm{tr}(X\Gamma^d Y\Gamma^c)(\Gamma^e)^\alpha{}_\delta(\Gamma^f)^\gamma{}_\beta. \tag{6.11}$$

7.1. Helicity and chirality

In four dimensions we define the chirality operator as

$$\gamma^5 := \mathring{\imath}\gamma^0\gamma^1\gamma^2\gamma^3. \tag{7.1}$$

a) Show that γ^5 satisfies

$$\{\gamma^5, \gamma^\mu\} = 0, \qquad \left(\gamma^5\right)^2 = \text{id}. \tag{7.2}$$

b) Show that the operators

$$P_{\text{R,L}} = \tfrac{1}{2}(\text{id} \pm \gamma^5), \tag{7.3}$$

are two orthogonal projectors to the chiral subspaces and that they satisfy the completeness relation

$$P_{\text{L}} + P_{\text{R}} = \text{id}. \tag{7.4}$$

c) Show that the Dirac Lagrangian $\mathcal{L} = \bar{\psi}(\gamma^\mu\partial_\mu - m)\psi$ is invariant under a chiral transformation $U = \exp(-\mathring{\imath}\alpha\gamma^5)$ of the fields for $m = 0$, and derive the associated conserved current. Show that a non-zero mass breaks the symmetry.

Helicity is defined to be the projection of spin along the direction of motion,

$$h(\vec{p}) = \frac{\vec{\Sigma}\cdot\vec{p}}{\|\vec{p}\|}\,. \tag{7.5}$$

Here, $\vec{\Sigma}$ is the spin operator which is given in the Weyl representation by

$$\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}. \tag{7.6}$$

d) Show that chirality is not conserved for a massive fermion by computing the equations of motions for the chiral fermions ψ_{L} and ψ_{R}, with

$$\psi_{\text{L,R}} = P_{\text{L,R}}\psi. \tag{7.7}$$

e) Show that helicity and chirality are equivalent for a massless spinor $u_\alpha(\vec{p})$.

f) Show that the Dirac equation respects helicity.

g) Argue that helicity is not Lorentz invariant for $m \neq 0$.

7.2. Gordon identity

Prove the Gordon identity,

$$\bar{u}_\beta(\vec{q})\gamma^\mu u_\alpha(\vec{p}) = \frac{\mathring{\imath}}{2m}\,\bar{u}_\beta(\vec{q})\left[(q+p)^\mu - \tfrac{1}{2}[\gamma^\mu, \gamma^\nu](q-p)_\nu\right]u_\alpha(\vec{p}). \tag{7.8}$$

Hint: You can do this using just $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\text{id}$ and the Dirac equation.

$\longrightarrow$

7.3. Spinors, spin sums and completeness relations

In this exercise we will use the Weyl representation specified in problem 6.1.

a) Show that $(p{\cdot}\sigma)(p{\cdot}\bar{\sigma}) = p^2\mathrm{id}$.

b) Prove that the below 4-spinor $u_\alpha(\vec{p})$ solves the Dirac equation $(\mathring{\imath}p_\mu\gamma^\mu - m)u_\alpha(\vec{p}) = 0$

$$u_\alpha(\vec{p}) = \begin{pmatrix} \sqrt{-p{\cdot}\sigma}\,\xi_\alpha \\ \mathring{\imath}\sqrt{p{\cdot}\bar{\sigma}}\,\xi_\alpha \end{pmatrix}, \tag{7.9}$$

where $\xi_\pm$ form a basis of 2-spinors.

c) Suppose, the 2-spinors ξ_+ and ξ_- are orthonormal. What does it imply for $\xi_\alpha^\dagger\xi_\beta$ and

$$\sum_{\alpha\in\{+,-\}} \xi_\alpha\xi_\alpha^\dagger\,? \tag{7.10}$$

d) Show that $\bar{u}_\alpha(\vec{p})u_\alpha(\vec{p}) = 2m$ for $\alpha \in \{+,-\}$.

e) Show the completeness relation:

$$\sum_{\alpha\in\{+,-\}} u_\alpha(\vec{p})\bar{u}_\alpha(\vec{p}) = \mathring{\imath}p_\mu\gamma^\mu + m. \tag{7.11}$$

7.4. Electrodynamics

Consider the Lagrange density for electrodynamics with an external source field J^μ

$$\mathcal{L}(A_\mu) = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu, \qquad \text{where} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{7.12}$$

a) Show that the Euler–Lagrange equations are the inhomogeneous Maxwell equations. The electric and magnetic fields are defined by $E_i = F_{0i}$ and $\varepsilon_{ijk}B_k = -F_{ij}$. What about the homogeneous Maxwell equations?

b) *for fun:* Show that all Maxwell equations can summarised in the spinorial equation

$$\gamma^\nu\gamma^\rho\gamma^\sigma\partial_\nu F_{\rho\sigma} = 2\gamma^\nu J_\nu. \tag{7.13}$$

For the remainder of this problem, we drop the sources, $J^\mu = 0$.

c) Construct the stress-energy tensor for this theory assuming that an infinitesimal translation δa^μ transforms the gauge field according to $\delta A_\rho = -\delta a^\nu\,\partial_\nu A_\rho$. Show that the resulting stress-energy tensor $T_0^{\mu\nu}$ is neither symmetric nor gauge invariant.

d) Now supplement the translation by a gauge transformation with gauge transformation parameter $\delta\alpha = \delta a^\nu\, A_\nu$ and compute the resulting stress-energy tensor $T^{\mu\nu}$. Convince yourself that the latter is symmetric and gauge invariant.

e) Show that the invariant stress-energy tensor $T^{\mu\nu}$ leads to the standard formulae for the electromagnetic energy and momentum densities

$$\mathcal{E} = \tfrac{1}{2}(\vec{E}^2 + \vec{B}^2), \qquad \vec{S} = \vec{E}\times\vec{B}. \tag{7.14}$$

8.1. The massive vector field

Consider the Lagrangian for the free massive vector field V_μ:

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu V^\nu \partial_\mu V_\nu + \tfrac{1}{2}\partial^\mu V^\nu \partial_\nu V_\mu - \tfrac{1}{2}m^2 V^\mu V_\mu. \tag{8.1}$$

a) Derive the Euler–Lagrange equations of motion for V_μ.

b) By taking a derivative of the equation, show that V_μ is a conserved current.

c) Show that V_μ satisfies the Klein–Gordon equation.

8.2. Hamiltonian formulation

The Hamiltonian formulation of the massive vector is somewhat tedious due to the presence of constraints. Let us consider the phase space.

a) Derive the momenta Π_μ conjugate to the fields V_μ. Considering the space and time components separately, what do you notice?

Your observation is related to constraints: the time component V_0 of the vector field is completely determined by V_k and Π_k (without making reference to time derivatives).

b) Use the equations derived in problem 8.1 to show that

$$V_0 = -m^{-2}\partial_k \Pi_k, \qquad \dot{V}_0 = \partial_k V_k. \tag{8.2}$$

c) Substitute this solution for V_0 and $\dot{V}_0$ into the Lagrangian and perform a Legendre transformation to obtain the Hamiltonian. Show that

$$H = \int d\vec{x}^3 \left(\tfrac{1}{2}\Pi_k\Pi_k + \tfrac{1}{2}m^{-2}\partial_k\Pi_k\partial_l\Pi_l + \tfrac{1}{2}\partial_k V_l \partial_k V_l - \tfrac{1}{2}\partial_l V_k \partial_k V_l + \tfrac{1}{2}m^2 V_k V_k\right). \tag{8.3}$$

d) Derive the Hamiltonian equations of motion for V_k and Π_k, and compare them to the results of problem 8.1.

8.3. Commutators

The unequal-time commutators $\Delta^{\mathrm{V}}_{\mu\nu}(x-y) = \mathring{\imath}[V_\mu(x), V_\nu(y)]$ for the massive vector field read

$$\Delta^{\mathrm{V}}_{\mu\nu}(x) = \left(\eta_{\mu\nu} - m^{-2}\partial_\mu\partial_\nu\right)\Delta(x), \tag{8.4}$$

where $\Delta(x)$ is the corresponding function for the scalar field.

a) Show that these obey the equations derived in problem 8.1.

b) Show explicitly that they respect the constraint equations in (8.2), i.e.

$$\left[m^2 V_0(x) + \partial_k \Pi_k(x), V_\nu(y)\right] = \left[\dot{V}_0(x) - \partial_k V_k(x), V_\nu(y)\right] = 0. \tag{8.5}$$

c) Confirm that the equal-time commutators take the canonical form

$$\left[V_k(\vec{x}), V_l(\vec{y})\right] = \left[\Pi_k(\vec{x}), \Pi_l(\vec{y})\right] = 0, \qquad \left[V_k(\vec{x}), \Pi_l(\vec{y})\right] = \mathring{\imath}\delta_{kl}\delta^3(\vec{x}-\vec{y}). \tag{8.6}$$

$\longrightarrow$

8.4. Photon propagator

The Lagrangian for the electromagnetic potential with gauge-fixing term reads

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}\xi^{-1}\big(\partial^\mu A_\mu\big)^2. \tag{8.7}$$

a) Show that the equations of motion for the photon field A_μ take the form $K^{\mu\nu}A_\nu = 0$ with the differential operator

$$K^{\mu\nu} := -\eta^{\mu\nu}\partial^2 + (1-\xi^{-1})\partial^\mu\partial^\nu. \tag{8.8}$$

b) Compute the Green function of the equation of motions defined by

$$K^{\mu\nu}G^{\mathrm{V}}_{\nu\rho}(x-y) = \delta^\mu_\rho\delta^4(x-y). \tag{8.9}$$

Hint: Transform the equation to momentum space and make a suitable ansatz for all conceivable Lorentz index structures in $G^{\mathrm{V}}_{\mu\nu}(p)$

8.5. Interacting scalar field theory and scale invariance

Consider the real scalar field with interactions

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{6}\kappa\phi^3 - \tfrac{1}{24}\lambda\phi^4. \tag{8.10}$$

a) The action $S = \int \mathrm{d}x^4\,\mathcal{L}$ is a dimensionless quantity. In natural units time and length have mass dimension $\mathrm{d}x \sim m^{-1}$. What are the mass dimensions of the field ϕ and the interaction couplings κ and λ?

b) The coordinate scaling transformation $x' = \Lambda x$ with some $\Lambda \in \mathbb{R}^+$ can be extended to the scalar field by $\phi'(x') = \Lambda^{-\Delta}\phi(x)$ for some $\Delta \in \mathbb{R}$. For which values of the parameters $\{m, \kappa, \lambda, \Delta\}$ is the action scale invariant?

From now on, we consider the scale invariant theory.

c) Let $\phi(x)$ be a solution to the equation of motion of this interacting theory, show that $\Lambda^\Delta\phi(\Lambda x)$ is also a solution.

d) Derive the scale (or dilatation) current using Noether's procedure for invariance under the scaling transformation defined in part b). Show explicitly that it is conserved.

The remainder of the problem deals with a suitable form for the stress-energy tensor in a scale invariant (or conformal) model.

e) Compute the stress-energy tensor and show that it has a non-vanishing trace. Use the equations of motion to write the trace in the form $T^\mu_\mu = \partial^2 K$.

f) Show that the scale current can be expressed as $S^\mu = x_\nu T^{\mu\nu} - \partial^\mu K$. Show that it is conserved using the properties of $T^{\mu\nu}$.

g) The stress-energy tensor can be improved by adding an extra term

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + c\big(\partial^\mu\partial^\nu K - \eta^{\mu\nu}\partial^2 K\big). \tag{8.11}$$

For which value of c is this tensor symmetric, conserved and traceless?

h) Show that the modified scale current $\tilde{S}^\mu = x_\nu\tilde{T}^{\mu\nu}$ is conserved and compare it to the original current S^μ.

9.1. Interaction picture

The field operator $\phi_0(x)$ in the interaction picture is related to the field operator $\phi(x)$ in the Heisenberg picture by

$$\phi(t,\vec{x}) = U(t)^{-1}\,\phi_0(t,\vec{x})\,U(t). \tag{9.1}$$

We assume the fields to coincide at the reference time t_0, $\phi(t_0) = \phi_0(t_0)$. The transformation operator is then given by

$$U(t) = \exp\bigl(\mathring{\imath}(t-t_0)H_0\bigr)\exp\bigl(-\mathring{\imath}(t-t_0)H\bigr). \tag{9.2}$$

a) Show that $U(t)$ satisfies the following differential equation and initial condition

$$\mathring{\imath}\,\frac{\partial}{\partial t}U(t) = H_{\text{int}}(t)\,U(t), \qquad U(t_0) = 1. \tag{9.3}$$

Determine the interaction Hamiltonian $H_{\text{int}}(t)$.

b) Show that the unique solution to this equation with the same initial condition and $t > t_0$ can be written as

$$U(t) = \mathrm{T}\exp\biggl(-\mathring{\imath}\int_{t_0}^{t}\mathrm{d}t'\,H_{\text{int}}(t')\biggr). \tag{9.4}$$

c) Show that the operator is unitary

$$U(t)^\dagger = U(t)^{-1}. \tag{9.5}$$

d) We define the time evolution operator $U(t_2,t_1)$ for $t_2 \geq t_1$ as the time-ordered exponential

$$U(t_2,t_1) := \mathrm{T}\exp\biggl(-\mathring{\imath}\int_{t_1}^{t_2}\mathrm{d}t\,H_{\text{int}}(t)\biggr), \tag{9.6}$$

while for $t_1 \geq t_2$ it is defined by $U(t_2,t_1) := U(t_1,t_2)^{-1}$.

Show that it satisfies the composition rule (for all permutations of the times t_k)

$$U(t_3,t_2)\,U(t_2,t_1) = U(t_3,t_1). \tag{9.7}$$

Deduce that it is related to $U(t)$ by $U(t_2,t_1) = U(t_2)U(t_1)^{-1}$.

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9.2. Feynman propagators

The Feynman propagator for the real scalar field is defined as

$$G_{\mathrm{F}}(y-x) = i\langle 0|\,\mathrm{T}\big[\phi(y)\phi(x)\big]\,|0\rangle = \begin{cases} i\langle 0|\phi(x)\phi(y)|0\rangle & \text{for } x^0 > y^0, \\ i\langle 0|\phi(y)\phi(x)|0\rangle & \text{for } y^0 > x^0, \end{cases} \tag{9.8}$$

where the time-ordering symbol T orders the fields within the product with decreasing times from left to right.

a) Show that the propagator satisfies the defining relation

$$(-\partial^2 + m^2)G_{\mathrm{F}}(x) = \delta^4(x). \tag{9.9}$$

Hint: write the propagator in terms of correlators and commutators.

The Feynman propagator for a Dirac field is defined by

$$G_{\mathrm{F}}^{\mathrm{D}a}{}_b(y-x) = i\langle 0|\,\mathrm{T}\big[\psi^a(x)\bar{\psi}_b(y)\big]\,|0\rangle = \begin{cases} i\langle 0|\psi^a(x)\bar{\psi}_b(y)|0\rangle & \text{for } x^0 > y^0, \\ -i\langle 0|\bar{\psi}_b(y)\psi^a(x)|0\rangle & \text{for } y^0 > x^0. \end{cases} \tag{9.10}$$

b) Write this propagator in terms of the Feynman propagator G_{F} for the real scalar field.

c) Show that the Feynman propagator for the Dirac field is a Green function of the Dirac equation:

$$(-\gamma{\cdot}\partial + m)^a{}_b G_{\mathrm{F}}^{\mathrm{D}b}{}_c(x) = \delta^a_c\delta^4(x). \tag{9.11}$$

d) Show that the Feynman propagator $G_{\mathrm{F}}(x)$ for scalars can be expressed as the following integral

$$G_{\mathrm{F}}(x) = \int_{\mathbb{R}^4} \frac{\mathrm{d}p^4}{(2\pi)^4}\,\frac{e^{ip\cdot x}}{p^2 + m^2 - i\epsilon}\,. \tag{9.12}$$

Derive a similar expression for the Feynman propagator for Dirac fields.

9.3. Wick's theorem

Wick's theorem relates the time-ordered product of fields $\phi_0(x)$ to the normal-ordered product plus all possible contractions

$$\mathrm{T}\big[\phi_0(x_1)\ldots\phi_0(x_m)\big] = \mathrm{N}\big[\phi_0(x_1)\ldots\phi_0(x_m) + \text{all contractions}\big]. \tag{9.13}$$

Prove this theorem by induction. What changes in the case of fermionic operators?

10.1. Four-point interaction in scalar QED

Consider a model of electrodynamics with a complex scalar field ϕ of charge q and mass m which couples to the vector potential A_μ. This model in the R_ξ-gauge is described by the Lagrangian density

$$\mathcal{L}_{\text{SQED}} = -(D^\mu\phi)^* D_\mu\phi - m^2\phi^*\phi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}\xi^{-1}(\partial^\mu A_\mu)^2, \tag{10.1}$$

where we set $\xi = 1$ for convenience. The covariant derivative D_μ and the electromagnetic field strength tensor $F_{\mu\nu}$ are given by

$$D_\mu = \partial_\mu - \mathring{\imath}qA_\mu, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{10.2}$$

In this exercise we want to compute perturbatively for small q the first non-trivial contribution to the time-ordered 4-point correlation function

$$\begin{aligned}&\langle 0|\,\mathrm{T}\big[\phi(x_1)\phi(x_2)\phi^\dagger(x_3)\phi^\dagger(x_4)\big]\,|0\rangle_{\text{int}}\\ &= \lim_{T\to\infty(1-\mathring{\imath}\epsilon)} \frac{\langle 0|\,\mathrm{T}\Big[\phi(x_1)\phi(x_2)\phi^\dagger(x_3)\phi^\dagger(x_4)\exp\Big(\mathring{\imath}\int_{-T}^{T}\mathrm{d}y^4\,\mathcal{L}_{\text{int}}(y)\Big)\Big]\,|0\rangle}{\langle 0|\,\mathrm{T}\Big[\exp\Big(\mathring{\imath}\int_{-T}^{T}\mathrm{d}y^4\,\mathcal{L}_{\text{int}}(y)\Big)\Big]\,|0\rangle}\,.\end{aligned} \tag{10.3}$$

a) Split the Lagrangian $\mathcal{L}$ of the model into a free part $\mathcal{L}_0$ and an interaction interaction Lagrangian $\mathcal{L}_{\text{int}}$.

b) Give a short explanation as to why we need to expand the 4-point correlation function (10.3) to second order in q to obtain the leading non-trivial contribution.

c) Expand the denominator of (10.3) to order q^2, then use Wick's theorem to decompose the time-ordered product into a sum of complete contractions between pairs of fields. Find a *pictorial* representation for the different contributions.

d) Expand the numerator of the time-ordered 4-point correlation function (10.3) in the same way. It may be useful to draw diagrams to simplify calculations.

e) Now combine the leading non-trivial contributions to (10.3) and group them according to the graph topology. How can you interpret the various contributions?

f) Consider the connected contributions to the 4-point correlation function. Insert the Feynman propagators of the scalar and photon fields

$$\begin{aligned} G_{\mathrm{F}}(x-y) &= \int \frac{\mathrm{d}p^4}{(2\pi)^4}\,\frac{\mathrm{e}^{\mathring{\imath}p\cdot(x-y)}}{p^2+m^2-\mathring{\imath}\epsilon}\,,\\ G_{\mathrm{F}}^{\mu\nu}(x-y) &= \int \frac{\mathrm{d}p^4}{(2\pi)^4}\,\frac{\eta^{\mu\nu}\,\mathrm{e}^{\mathring{\imath}p\cdot(x-y)}}{p^2-\mathring{\imath}\epsilon}\,,\end{aligned} \tag{10.4}$$

and perform all elementary integrations. How do you interpret the individual terms in the result?

g) How do you interpret the limit $T\to\infty$ in (10.3)?

h) How will your result in part f) change if you use a different gauge? E.g. use the R_ξ-gauge with $\xi \neq 1$.
Hint: The gauge only affects the photon propagator $G_{\mathrm{F}}^{\mu\nu}$.

11.1. Møller scattering

a) Calculate the $\mathcal{O}(q^2)$ leading connected contribution to the scattering matrix element M for Møller scattering,

$$e^-(p_1,\alpha) + e^-(p_2,\beta) \longrightarrow e^-(q_1,\gamma) + e^-(q_2,\delta), \tag{11.1}$$

through direct evaluation in position space. In this process the spins of the electrons are denoted by $\alpha,\ldots,\delta$; p_i and q_i denote the momenta of the particles.

b) Repeat the calculation in part a) using the Feynman rules for QED in momentum space.

11.2. Compton scattering

a) Calculate the $\mathcal{O}(q^2)$ leading connected contribution to the scattering matrix element M for Compton scattering,

$$e^-(p_1,\alpha) + \gamma(p_2,\sigma) \to e^-(q_1,\beta) + \gamma(q_2,\rho), \tag{11.2}$$

through direct evaluation in position space. In this process the spins of the electrons are denoted by α and β, while ρ and σ denote the polarisations of the photons; p_i and q_i denote the momenta of the particles. What changes if we replace the electrons by positrons?

b) Repeat the calculation in part a) using the Feynman rules for QED in momentum space.

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11.3. Kinematics in $2 \to 2$ scattering

Consider a $2 \to 2$ particle scattering process with the kinematics $p_1 + p_2 \to q_1 + q_2$ and masses $m_1, \ldots, m_4$.

a) Show that in the centre-of-mass frame, the energies $e(\vec{p}_i)$, $e(\vec{q}_i)$ and the norms of momenta $\|\vec{p}_i\|$, $\|\vec{q}_i\|$ of the incoming and the outgoing particles are entirely fixed by the total centre-of-mass energy $\sqrt{s}$ and the particle masses m_i.

b) Show that the scattering angle θ between $\vec{p}_1$ and $\vec{q}_1$ is given by

$$\theta = \arccos\left(\frac{s(t-u) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\sqrt{\lambda(s, m_1^2, m_2^2)}\,\sqrt{\lambda(s, m_3^2, m_4^2)}}\right), \tag{11.3}$$

with the Mandelstam variables given by

$$s = -(p_1 + p_2)^2, \qquad t = -(p_1 - q_1)^2, \qquad u = -(p_1 - q_2)^2, \tag{11.4}$$

and the Källén function defined as

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \tag{11.5}$$

c) Show that $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$.

d) Determine t_{min} and t_{max} from the condition $|\cos\theta| \leq 1$, and study the behaviour of t_{min} and t_{max} in the limit $s \gg m_i^2$.

e) Show that the general expression for the differential scattering cross section for $2 \to 2$ particle scattering

$$\mathrm{d}^6\sigma = \frac{(2\pi)^4\delta^4(p_1 + p_2 - q_1 - q_2)}{4\big\|e_1(\vec{p}_1)\vec{p}_2 - e_2(\vec{p}_2)\vec{p}_1\big\|}\,\frac{\mathrm{d}\vec{q}_1^{\,3}}{(2\pi)^3\,2e_3(\vec{q}_1)}\,\frac{\mathrm{d}\vec{q}_2^{\,3}}{(2\pi)^3\,2e_4(\vec{q}_2)}\,|M|^2 \tag{11.6}$$

reduces in the centre-of-mass frame to the following expression upon elimination of the momenta fixed by momentum conservation

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}^2\Omega} = \frac{\|\vec{q}_1\|}{\|\vec{p}_1\|}\,\frac{|M|^2}{64\pi^2 s}\,. \tag{11.7}$$

Furthermore, show that for equal masses $m_i = m$, it reduces further to

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}^2\Omega} = \frac{|M|^2}{64\pi^2 s}\,. \tag{11.8}$$

12.1. Optical theorem

For the non-trivial part of the scattering matrix we have that

$$\langle f|T|i\rangle = (2\pi)^4 \delta^4(P_i - P_f) M_{fi}, \tag{12.1}$$

where $S = 1 + \mathring{\imath} T$. Let us derive the optical theorem.

a) Use unitarity of the S-matrix to show that $T^\dagger T = -\mathring{\imath}(T - T^\dagger)$.

b) From this derive the optical theorem

$$M_{fi} - M_{if}^* = \mathring{\imath} \sum_X \int \mathrm{d}\mathit{\Pi}_X \, (2\pi)^4 \delta^4(P_f - P_X) M_{Xi} M_{Xf}^*. \tag{12.2}$$

Hint: Use the completeness relation $1 = \sum_X \int \mathrm{d}\mathit{\Pi}_X \, |X\rangle\langle X|$ with

$$\mathrm{d}\mathit{\Pi}_X = \prod_{j \in X} \frac{\mathrm{d}\vec{k}_j^3}{(2\pi)^3 \, 2e(\vec{k}_j)} \,. \tag{12.3}$$

c) Specialise to the case $|i\rangle = |f\rangle = |A\rangle$ where A is a two-particle state. Use the optical theorem to derive an expression for the total cross section $\sigma_{\text{tot}} = \sum_X \sigma_{XA}$ in terms of the imaginary part of the forward scattering amplitude $\operatorname{Im} M_{AA}$.

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12.2. Muon pair production

Follow the steps below to calculate the total cross section for muon pair production $e^-(p_1)e^+(p_2) \to \mu^-(q_1)\mu^+(q_2)$. This process is described by the QED Lagrangian with two fermion flavours $f = e, \mu$ with different masses but the same charge. The Lagrangian therefore reads

$$\mathcal{L} = \sum_{f=e,\mu} \bar{\psi}_f(\gamma^\mu D_\mu - m_f)\psi_f - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{12.4}$$

with $D_\mu = \partial_\mu - \mathring{\imath} q A_\mu$.

a) Draw all the connected diagrams that contribute to this process at the lowest non-trivial order, and use the Feynman rules for QED in momentum space to obtain the scattering amplitude M.

b) Compute $|M|^2$. Assuming that the particle spins are not measured, sum over the spins of the outgoing particle, and average over those of the incoming ones. This should help you bring your expression for $|M|^2$ into a much simpler form.

Hint: You might find the completeness relations for spinors useful.

c) The differential cross section in the centre-of-mass frame is given by

$$\mathrm{d}^6\sigma = \frac{|M|^2}{4\|\vec{p}_1\|\sqrt{s}} \frac{\mathrm{d}\vec{q}_1{}^3}{(2\pi)^3\, 2e_\mu(\vec{q}_1)} \frac{\mathrm{d}\vec{q}_2{}^3}{(2\pi)^3\, 2e_\mu(\vec{q}_2)} (2\pi)^4\delta^4(p_1 + p_2 - q_1 - q_2). \tag{12.5}$$

Use the result for $|M|^2$ obtained in part b), and integrate over $\vec{q}_1$ and $\vec{q}_2$ to obtain the total cross section $\sigma = \int \mathrm{d}^6\sigma$.

12.3. Polarised muon-electron scattering

Let us study the process $e^-(p_1)\mu^-(p_2) \to e^-(q_1)\mu^-(q_2)$, i.e. the scattering of negative muons on electrons. To simplify the calculation we assume both the electron and the muon to be massless. This is a good approximation for sufficiently high beam energies.

a) As a first step, calculate $|M|^2$ for unpolarised scattering (averaging over incoming particle spins). We assume that the spins of the outgoing particles are not measured. How is this result related to the one for muon pair production of problem 12.2?

b) We now assume the incoming muon beam to be fully polarised along its direction of motion. How does the result change compared to the completely unpolarised case? Can you predict or explain the outcome using parity invariance of QED?

Hint: Recall that the helicity and chiral eigenstates coincide in the massless case. You can therefore make use of the chiral projectors $P_{\mathrm{R,L}} = \frac{1}{2}(1 \pm \gamma^5)$ for $h_\mu = \pm 1/2$.

c) Consider the cases of electrons and muons having equal and opposite polarisations and calculate the additional contributions. How do the results for $|M|^2$ depend on the angle θ in the centre-of-mass frame? Can you identify special angles?

Quantum Field Theory I

Problem Set 13

ETH Zurich, 2025 HS

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13.1. Feynman and Schwinger parameters

Most loop integrals from Feynman diagrams are not immediately of the form advertised in problem 13.3. In order to evaluate them, one has to combine the propagator factors in the denominator. This can be achieved by introducing auxiliary integrals over so-called *Schwinger parameters* and *Feynman parameters*.

a) Prove the Schwinger parametrisation:

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty \mathrm{d}\alpha\, \alpha^{\nu-1}\, \mathrm{e}^{-\alpha A}\,. \tag{13.1}$$

Argue how it can be used to combine denominators of Feynman diagrams such that the momentum integrals can be performed efficiently.

b) The basic version of the Feynman parameter integral is

$$\frac{1}{AB} = \int_0^1 \frac{\mathrm{d}x}{[xA + (1-x)B]^2}\,. \tag{13.2}$$

Prove the generalisation to n propagator factors A_i raised to an arbitrary power ν_i by recursion:

$$\frac{1}{\prod_{i=1}^n A_i^{\nu_i}} = \frac{\Gamma(\sum_{i=1}^n \nu_i)}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^1 \left(\prod_{i=1}^n \mathrm{d}x_i\right) \delta\left(1 - \sum_{i=1}^n x_i\right) \frac{\prod_{i=1}^n x_i^{\nu_i - 1}}{[\sum_{i=1}^n x_i A_i]^{\sum_{i=1}^n \nu_i}}\,. \tag{13.3}$$

Hint: Consider the replacements $x_i = y_i/(1+y_n)$ and $y_n = z\sum_{i=1}^{n-1} y_i A_i/A_n$.

13.2. Volume of spheres

The integrands of D-dimensional euclidean integrals for Feynman diagrams can be brought to a spherically symmetric form $F(\ell_\mathrm{E}) = F(\|\ell_\mathrm{E}\|)$ using the results of problem 13.1. The angular part of the integral yields the volume of the $(D-1)$-dimensional sphere S^{D-1}. Use the Gaußian integral

$$\int_{-\infty}^\infty \mathrm{d}x \exp(-x^2) = \sqrt{\pi} \tag{13.4}$$

to derive the volume of the $(D-1)$-sphere:

$$\mathrm{Vol}(\mathrm{S}^{D-1}) = \frac{2\pi^{D/2}}{\Gamma(D/2)}\,. \tag{13.5}$$

Hint: Perform the Gaußian integral generalised to D dimensions in cartesian and spherical coordinates and compare the results.

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13.3. Basic loop integral

When computing Feynman diagrams in momentum space, one encounters the following basic loop integral in D-dimensional Minkowski space:

$$I_n(\mu^2) := \int \frac{-\mathring{\imath}\,\mathrm{d}\ell^D}{(2\pi)^D} \frac{1}{(\ell^2+\mu^2)^n}\,. \tag{13.6}$$

Here, n is a positive integer and $\mu = \mu(p_i)$ is an effective mass that is a scalar function of the external momenta p_i but independent of the loop momentum ℓ.

a) Perform a Wick rotation to euclidean space and transform the integral to spherical coordinates. Carry out the angular integration using spherical symmetry.

b) Find a criterion for divergence of the above integral in the UV region ($\|\ell_{\mathrm{E}}\| \to \infty$). For which values of n is the integral divergent in $D = 4$ dimensions?

c) Perform the radial integral to show that:

$$I_n(\mu^2) = \frac{\mu^{D-2n}}{(4\pi)^{D/2}} \frac{\Gamma(n-D/2)}{\Gamma(n)}\,. \tag{13.7}$$

d) Demonstrate that the derivative $(\mathrm{d}/\mathrm{d}\mu^2) I_2(\mu^2)$ in $D = 4$ dimensions is finite. Use this feature to show that the subtracted loop integral I_2 can be written as

$$I_2(\mu^2) - I_2(\mu_0^2) = -\frac{1}{16\pi^2} \log \frac{\mu^2}{\mu_0^2}\,. \tag{13.8}$$

13.4. Loop integral numerators

More elaborate loop integrals can carry some loop momenta ℓ^μ in the numerator, e.g.:

$$I_n^\mu(\mu^2) := \int \frac{-\mathring{\imath}\,\mathrm{d}\ell^D}{(2\pi)^D} \frac{\ell^\mu}{(\ell^2+\mu^2)^n}\,, \qquad I_n^{\mu\nu}(\mu^2) := \int \frac{-\mathring{\imath}\,\mathrm{d}\ell^D}{(2\pi)^D} \frac{\ell^\mu \ell^\nu}{(\ell^2+\mu^2)^n}\,. \tag{13.9}$$

Relate these integrals to the scalar integral of problem 13.3 to show that:

$$I_n^\mu(\mu^2) = 0, \qquad I_n^{\mu\nu}(\mu^2) = \frac{\eta^{\mu\nu}}{2} \frac{\mu^{D-2n+2}}{(4\pi)^{D/2}} \frac{\Gamma(n-1-D/2)}{\Gamma(n)}\,. \tag{13.10}$$

13.5. Dimensional regularisation

The UV behaviour of a loop integral depends on its dimensionality D.

a) For which values of n is the integral I_n in problem 13.3 UV divergent for $D = 3$ dimensions? In how many dimensions is the integral I_1 UV finite?

In the dimensional regularisation scheme, the dimension of the momentum loop integration is shifted from $D = 4$ to a suitable number to make the integral finite. Of course, the physical result eventually requires the correct dimensionality for the loop integrals. The crucial idea is to generalise D to be a complex number rather than an integer, and assume the results of problems 13.2, 13.3 and 13.4 to remain valid as functions of D. We can then analytically continue the integrals back to their physical dimension in the complex plane and thus effectively use the dimensionality D of spacetime as a regulator.

b) Argue why we can use D as a regulator for the class of loop integrals in problems 13.3 and 13.4. Can you say how divergences manifest in this scheme?